VIII. SUPERSYMMETRIC GAUGE THEORIES

In this section we will show how we can construct theories which satisfy both supersymmetry and gauge invariance.

As a first step we will try to introduce a local symmetry in a theory which contains chiral multiplets. This will give us the possibility to understand how to put together a gauge symmetry and supersymmetry and will show us the basic ingredients we need to build a gauge theory, namely the introduction of a new kind of superfield that corresponds to the gauge multiplet representation of $N=1$ susy.

For simplicity in this section we will consider only $N=1$ theories, moreover we will assume that the gauge generators <u>commute</u> with the susy generators Q. Notice that for simple supersymmetry this is a mild assumption, given that there is just one susy generator Q_α, which can only furnish a trivial representation of any semi-simple gauge group. The only symmetry which can act non-trivially on the Q's are Abelian symmetries (recall the R-symmetry).

<u>Gauge invariant actions for chiral superfields</u>

If the gauge symmetry commutes with the Q generators, as we assumed, then each component field in a supermultiplet must transform in the same way under a gauge transformation. For a chiral superfield

$$\begin{cases} z_n(x) \to \sum_m \left[\exp\left(i \sum_A t^A \lambda^A(x)\right)\right]_{nm} z_m(x) \\ \psi_n(x) \to \sum_m \left[\exp\left(i \sum_A t^A \lambda^A(x)\right)\right]_{nm} \psi_m(x) \\ F_n(x) \to \sum_m \left[\exp\left(i \sum_A t^A \lambda^A(x)\right)\right]_{nm} F_m(x) \end{cases} ,$$

where t^A are Hermitian matrices representing the generators of the gauge group and $\lambda^A(x)$ are <u>real</u> functions of x^μ that parametrize a finite gauge transformation.

To use the superfield formalism we need to generalize the above transformation to find a corresponding superfield transformation. The main difficulty comes from the fact that a chiral superfield contains some terms which are given by derivatives of the component fields (the $\theta\sigma^\mu\bar\theta$, the $\theta\theta\bar\theta$ and the $\theta\theta\bar\theta\bar\theta$ components), so we can not just apply the gauge transformation rule we used for the z, ψ and F components to the whole superfield (recall that $\lambda^A(x)$ depends on x^μ!).

However we can use a shortcut to derive the correct form of the transformation. We notice that a chiral superfield rewritten in terms of the $y^\mu = x^\mu + i\theta\sigma^\mu\bar\theta$ and θ coordinates do not contain derivatives:

$$\Phi_m(y,\theta) = z_m(y) + \sqrt{2}\,\theta\,\psi_m(y) - \theta\theta\,F_m(y).$$

This implies that its transformation properties must be

$$\Phi_n(x,\theta,\bar\theta) = \Phi_n(y,\theta) \to \sum_m \left[\exp\left(i\sum_A t^A \lambda^A(y)\right)\right]_{nm} \Phi_m(y,\theta) \quad , \qquad (*)$$

where the gauge parameters now depend on y^μ and not just x^μ.

If a term in the action depends only on chiral superfields, and not their derivatives or complex conjugates, like the superpotential term $\int d^2\theta\, W(\Phi)$, then it will be invariant under the <u>local</u> transformation $(*)$, if it is invariant under <u>global</u> transformations with $\lambda^A(x)$ independent of x^μ.

However a term which contains $\bar\Phi$ as well as Φ does not have the same property and will force us to introduce gauge superfields in order to make it gauge invariant. A term of this kind is the Kähler potential. Let's see closer where the problem originates.

Because the matrices t^A are Hermitian we get that $(*)$ implies

$$\bar\Phi_n(x,\theta,\bar\theta) = \bar\Phi_n(y^+,\bar\theta) \to \sum_m \bar\Phi_m(y^+,\bar\theta)\left[\exp\left(-i\sum_A t^A (\lambda^A(y))^*\right)\right]_{mn}$$

$$= \sum_m \bar\Phi_m(y^+,\bar\theta)\left[\exp\left(-i\sum_A t^A\lambda^A(y^+)\right)\right]_{mn}.$$

In general $(\lambda^A(y))^* = \lambda^A(y^+)$ and $\lambda^A(y)$ are different, so $\bar\Phi$ does not transform with the inverse of the transformation matrix for Φ. This means that $\bar\Phi\Phi$ is <u>not</u> gauge invariant, hence the Kähler potential is not gauge invariant if we do not modify it.

To solve the problem we must introduce a gauge connection matrix $\Gamma_{mn}(x,\theta,\bar\theta)$, with the transformation property

$$\Gamma(x,\theta,\bar\theta) \to \exp\left(+i\sum_A t^A\lambda^A(y^+)\right)\Gamma(x,\theta,\bar\theta)\exp\left(-i\sum_A t^A\lambda^A(y)\right). \qquad (**)$$

By multiplying $\bar\Phi$ on the right with Γ we get a superfield that transforms as

$$\left[\bar\Phi(x,\theta,\bar\theta)\,\Gamma(x,\theta,\bar\theta)\right]_n \to \sum_m \left[\bar\Phi(x,\theta,\bar\theta)\,\Gamma(x,\theta,\bar\theta)\right]_m \left[\exp\left(-i\sum_A t^A\lambda^A(y)\right)\right]_{mn} ,$$

so any <u>globally</u> gauge-invariant function constructed from Φ and $\bar\Phi\Gamma$ (and <u>not</u> their derivatives or complex conjugates) will also be <u>locally</u> gauge-invariant.

One obvious example is the gauge-invariant version $(\bar\Phi\Gamma\Phi)_D$ of the D-term in the renormalizable Lagrangian for a chiral superfield.

Any $\Gamma(x,\theta,\bar\theta)$ that transforms as in (**) will allow us to construct gauge-invariant Lagrangians of chiral superfields. The choice is not unique: if we multiply Γ on the right with a chiral superfield Y with the transformation rule

$$Y \to \exp\left(i\sum_A t^A \lambda^A(x)\right) Y \exp\left(-i\sum_A t^A \lambda^A(x)\right),$$

then we get a new gauge connection that also satisfies (**).

· The N=1 Vector superfield

In the following we will use the freedom in the choice of Γ to find a simple form for the gauge connection and to identify the gauge multiplet related to the gauge symmetry.

A first simplification is to take Γ to be Hermitian

$$\Gamma^+(x,\theta,\bar\theta) = \Gamma(x,\theta,\bar\theta).$$

This is always possible because if Γ satisfy the transformation rule (**), then also Γ^+ transforms according to (**). This means that if Γ is not Hermitian, we can replace it with its Hermitian part $(\Gamma+\Gamma^+)/2$ (or, if that vanishes, with its anti-Hermitian part $(\Gamma-\Gamma^+)/2i$).

Another simplification is to express Γ in terms of fields whose transformation properties are independent of the specific representation t^A of the gauge algebra. For this purpose we take Γ in the form

$$\Gamma(x,\theta,\bar\theta) = \exp\left(2\sum_A t^A V^A(x,\theta,\bar\theta)\right),$$

where $V^A(x,\theta,\bar\theta)$ are a set of <u>real superfields</u> (so that Γ is Hermitian), not depending on the representation of the gauge algebra furnished by the t^A. This can be understood by using the Baker-Hausdorff formula

$$e^A e^B = \exp\left(a+b+\tfrac{1}{2}[a,b]+\tfrac{1}{12}[a,[a,b]]+\tfrac{1}{12}[b,[b,a]]+\cdots\right)$$

which implies

$$\exp\left(\sum_A a^A t^A\right)\exp\left(\sum_A b^A t^A\right) = \exp\left(\sum_A g^A(a,b)\,t^A\right),$$

where

$$g^A(a,b) = a^A + b^A + \tfrac{1}{2}i\sum_{BC} f^A{}_{BC}\, a^B b^C - \frac{1}{12}\sum_{BCDE} f^A{}_{BC} f^C{}_{DE}\, a^B a^D b^E - \frac{1}{12}\sum_{BCDE} f^A{}_{BC} f^C{}_{DE}\, b^B b^D a^E + \cdots$$

which depends only on the Lie algebra

$$[t^B, t^C] = i\sum_A f^A{}_{BC}\, t^A,$$

but not on the particular representation furnished by the t^A.

We can obtain a further simplification by noting an additional symmetry of supersymmetric gauge theories. If a function of Φ and $\bar{\Phi}\Gamma$ is invariant under global gauge transformations, then it will automatically be invariant not only under the local gauge transformations, but also under the larger group of extended gauge transformations

$$\Phi_n(x,\theta,\bar\theta) \to \sum_m \left[\exp\left(-i\sum_A t^A \Lambda^A(x,\theta,\bar\theta)\right)\right]_{nm} \Phi_m(x,\theta,\bar\theta)$$

and

$$\Gamma(x,\theta,\bar\theta) \to \exp\left(-i\sum_A t^A \bar\Lambda^A(x,\theta,\bar\theta)\right)\Gamma(x,\theta,\bar\theta)\exp\left(i\sum_A t^A \Lambda^A(x,\theta,\bar\theta)\right) ,$$

where $\Lambda^A(x,\theta,\bar\theta)$ is an arbitrary chiral superfield. Under this transformation

$$V^A(x,\theta,\bar\theta) \to V^A(x,\theta,\bar\theta) + \frac{i}{2}\left[\Lambda^A(x,\theta,\bar\theta) - \bar\Lambda^A(x,\theta,\bar\theta)\right] + \cdots , \qquad (***)$$

where "..." denotes terms arising from the commutators in the Baker-Hausdorff formula, which are of first or higher order in the gauge coupling constants.

Let's expand $V^A(x,\theta,\bar\theta)$ in components. We recall that we chose the condition $V^+ = V$, so we get

$$V^A(x,\theta,\bar\theta) = f^A(x) + \theta\chi^A(x) + \bar\theta\bar\chi^A(x) + \theta\theta\, m^A(x) + \bar\theta\bar\theta\, m^{A+}(x)$$
$$+\theta\sigma^\mu\bar\theta\, v^A_\mu + i(\theta\theta)\bar\theta\left(\bar\lambda^A + \frac{1}{2}\bar\sigma^\mu\partial_\mu\chi^A\right) - i(\bar\theta\bar\theta)\theta\left(\lambda^A - \frac{1}{2}\sigma^\mu\partial_\mu\bar\chi^A\right)$$
$$+\frac{1}{2}(\theta\theta)(\bar\theta\bar\theta)\left(D^A - \frac{1}{2}\Box f^A\right).$$

Notice that in the above expansion we changed notation with respect to the previously used expansion of the general scalar superfield. In particular we redefined the cubic and quartic terms in the θ expansion by subtracting suitable functions of χ and f. This will be useful to simplify the notation afterwards.

To find the effect of an extended susy transformation on the vector superfield components we also need the expansion of the chiral superfield $\Lambda^A(x,\theta,\bar\theta)$. We find

$$\frac{1}{2}i\left(\Lambda^A - \bar\Lambda^A\right) = \frac{i}{2}(z^A - z^{A+}) + i\frac{\sqrt2}{2}\theta\psi^A - i\frac{\sqrt2}{2}\bar\theta\bar\psi^A - \frac{i}{2}\theta\theta F^A + \frac{i}{2}\bar\theta\bar\theta F^{A+}$$
$$-\frac{1}{2}\theta\sigma^\mu\bar\theta\,\partial_\mu(z^A + z^{A+}) - \frac{1}{2\sqrt2}\theta\theta\,\partial_\mu\psi^A\sigma^\mu\bar\theta - \frac{1}{2\sqrt2}\bar\theta\bar\theta\,\partial_\mu\bar\psi^A\bar\sigma^\mu\theta$$
$$-i\frac{1}{8}\theta\theta\bar\theta\bar\theta\,\Box(z^A - z^{A+}).$$

Substituting the explicit expressions of V^A and Λ^A into $(***)$ we find the following transformation rules

$$f^A \to f^A - \mathrm{Im}\, z^A + \cdots$$
$$\chi^A \to \chi^A + \frac{i}{\sqrt2}\psi^A + \cdots$$

$$m^A \to m^A - \frac{i}{2} F^A + \cdots$$

$$v_\mu{}^A \to v_\mu{}^A - \partial_\mu \operatorname{Re} \Xi^A + \cdots$$

$$\lambda^A \to \lambda^A + \cdots$$

$$D^A \to D^A + \cdots$$

where "$\cdots$" denotes terms that arise from the structure constants of the gauge algebra and are therefore proportional to one or more factors of gauge coupling constants.

We can use such an extended gauge transformation to put the gauge superfields into a convenient form, known as <u>Wess–Zumino gauge</u> in which

$$\underline{f^A = \chi^A = m^A = 0},$$

so that

$$\underline{V^A_{WZ}(x,\theta,\bar\theta) = \theta\sigma^\mu\bar\theta\, v_\mu{}^A + i(\theta\theta)\bar\theta\bar\lambda^A - i(\bar\theta\bar\theta)\theta\lambda^A + \frac{1}{2}(\theta\theta)(\bar\theta\bar\theta)D^A}.$$

To do this at zero order in the coupling constants we just need to choose

$$\begin{cases} \operatorname{Im} \Xi^A = f^A \\ \psi^A = +\sqrt{2}\, i \chi^A \\ F^A = -2 i m^A \end{cases}.$$

For Abelian gauge theories this is enough because the structure constants of the group vanish, so one has no extra terms in the transformation rules for the V^A component fields.

For non-Abelian theories one needs to cancel the f, χ and m components taking into account also the terms in the transformation rules which depend on the structure constants. Thus the actual values of $\operatorname{Im}\Xi^A$, ψ^A and F^A needed to reach the Wess–Zumino gauge will be power series in the gauge coupling constants. There is however no need to find the complete form of the transformation needed to reach the Wess–Zumino gauge: the important thing is that it is possible to find such a transformation.

The Wess–Zumino gauge condition is not invariant under supersymmetry transformations (unless $v^A_\mu = \lambda^A = 0$, and the condition $\lambda^A = 0$ is not supersymmetric unless also $D^A = 0$, in which case the whole superfield vanishes). Once we adopt the Wess–Zumino gauge, the action is no longer invariant under either general extended gauge transformations or under supersymmetry. However it is invariant under supersymmetry transformations, which take us out of the Wess–Zumino gauge, followed by a suitable extended gauge transformation that takes us back to the Wess–Zumino gauge.

· Gauge transformations

Now we will investigate the transformations of the vector superfield under ordinary gauge transformations. First of all we notice that there are some residual gauge symmetries which are not fixed by the Wess-Zumino gauge: they are given by the transformations with Λ^A given by chiral superfields satisfying

$$\begin{cases} \psi^A = 0 \\ F^A = 0 \\ z^A = z^{A+} \end{cases},$$

or, explicitly

$$\begin{cases} \Lambda^A + \bar\Lambda^A = (z^A + z^{A+}) - \frac{1}{4}(\theta\theta)(\bar\theta\bar\theta)\,\Box(z^A + z^{A+}) = 2z^A - \frac{1}{2}(\theta\theta)(\bar\theta\bar\theta)\,\Box z^A \\ \Lambda^A - \bar\Lambda^A = i\,\theta\sigma^\mu\bar\theta\,\partial_\mu(z^A + z^{A+}) = 2i\,\theta\sigma^\mu\bar\theta\,\partial_\mu z^A \end{cases}$$

To find the infinitesimal transformation of V^A, we can apply the Baker-Hausdorff formula. To first order in Λ^A we get

$$\delta V = \frac{i}{2} L_V(\Lambda + \bar\Lambda) + \frac{i}{2} L_V \coth L_V(\Lambda - \bar\Lambda),$$

where we defined

$$V \equiv \sum_A t^A V^A$$
$$\Lambda \equiv \sum_A t^A \Lambda^A$$

and $L_X Y$ denotes the Lie derivative

$$L_X Y = [X, Y],$$
$$(L_X)^2 Y = [X,[X,Y]],$$
$$\vdots$$

The term $L_{V/2}\coth L_{V/2}$ is meant to be expanded in series

$$x\coth x = 1 + \frac{x^2}{3} - \frac{x^4}{45} + \dots$$

The transformation rule becomes much simpler in the Wess-Zumino gauge. Given that all the terms in V_{WZ} contain at least two powers of θ only terms with at most two powers of V_{WZ} survive in the expansion. Moreover, with the previously defined Λ, $\Lambda - \bar\Lambda$ already contains at least two powers of θ, so only the constant term survives in the expansion of $x\coth x$. Thus we get for an infinitesimal gauge transformation

$$\underline{\delta V_{WZ} = \frac{i}{2}(\Lambda - \bar\Lambda) - \frac{i}{2}\left[(\Lambda + \bar\Lambda), V_{WZ}\right].}$$

This transformation gives the usual non-Abelian gauge transformations for the component fields:

$$\begin{cases} \delta v_\mu{}^A = -\partial_\mu \xi^A - f^A{}_{BC}\, v_\mu{}^B \xi^C \\ \delta \lambda^A = -f^A{}_{BC}\, \lambda^B \xi^C \\ \delta D^A = -f^A{}_{BC}\, D^B \xi^C \end{cases} .$$

We can see that $v_\mu{}^A$ is the non-Abelian gauge field. λ^A are a set of fermions which transform in the adjoint representation of the gauge group (they are called gaugino fields). D^A are real scalars again in the adjoint representation of the gauge group and they will turn out to be a set of auxiliary fields.

• To construct gauge invariant actions with chiral superfields we need to know explicitly the expression for $\Gamma(x,\theta,\bar\theta)$. Fortunately, in the Wess-Zumino gauge, we can easily compute $\Gamma(x,\theta,\bar\theta)$ by expanding in series of V_{WZ}, the particular form of V_{WZ} ensures that terms containing V_{WZ}^3 are vanishing (they would contain at least 6 factors of θ), so we are left with the first three terms in the expansion

$$\underline{\Gamma(x,\theta,\bar\theta) = \exp(2V_{WZ}) = 1 + 2V_{WZ} + 2V_{WZ}^2}\ .$$

We find

$$V_{WZ}^2 = \frac{1}{2}(\theta\theta)(\bar\theta\bar\theta) \sum_{A,B} t^A t^B\, v^{A\mu} v^B{}_\mu\ .$$

We can use this result to compute the gauge-invariant version of the renormalizable Kähler potential for chiral superfields. Writing only the terms which contribute to the D-term we get (we use the notation $v_\mu \equiv \sum_A t^A v_\mu{}^A$)

$$\bar\Phi V_{WZ} \Phi\big|_{\theta\theta\bar\theta\bar\theta} = \frac{i}{2} z^+ v^\mu \partial_\mu z - \frac{i}{2}\partial_\mu z^+ v^\mu z + \frac{1}{2} z^+ D z - \frac{1}{2}\bar\psi\bar\sigma^\mu v_\mu \psi + \frac{\sqrt2}{2} i\,(z^+\lambda\psi - \bar\psi\bar\lambda z)\ ,$$

$$\bar\Phi V_{WZ}^2 \Phi\big|_{\theta\theta\bar\theta\bar\theta} = \frac{1}{2} z^+ v^\mu v_\mu z\ .$$

Putting the various pieces together we get

$$\bar\Phi e^{2V_{WZ}} \Phi\big|_{\theta\theta\bar\theta\bar\theta} = (D_\mu z)^+ D^\mu z - i\bar\psi\bar\sigma^\mu D_\mu\psi + F^+F + \sqrt2\, i\, z^+\lambda\psi - \sqrt2\, i\,\bar\psi\bar\lambda z + z^+ D z + \text{total derivative},$$

where we defined the gauge covariant derivatives

$$\begin{cases} D_\mu z = \partial_\mu z - i\, v_\mu z \\ D_\mu \psi = \partial_\mu\psi - i\, v_\mu \psi \end{cases} .$$

· Gauge invariant action for Abelian gauge superfields

We want now construct a gauge-invariant supersymmetric action for the gauge superfields. We will first consider the Abelian case and then we will generalize it to the non-Abelian gauge symmetries.

In an Abelian gauge theory the gauge-invariant field constructed from v_μ is the field-strength tensor

$$f_{\mu\nu}(x) = \partial_\mu v_\nu(x) - \partial_\nu v_\mu(x).$$

This will be the starting point to find a supersymmetric generalization, that is a supermultiplet which contains the field-strength as a component field. The transformation rule of $f_{\mu\nu}$ under a susy transformation is

$$\delta f_{\mu\nu} = \varepsilon(\sigma_\mu \partial_\nu - \sigma_\nu \partial_\mu)\bar\lambda - \bar\varepsilon(\bar\sigma_\mu \partial_\nu - \bar\sigma_\nu \partial_\mu)\lambda.$$

The transformation for λ is

$$\delta\lambda = i\varepsilon D - \sigma^{\mu\nu}\varepsilon f_{\mu\nu} ,$$

and the one for D is

$$\delta D = -\partial_\mu(\lambda\sigma^\mu\bar\varepsilon + \varepsilon\sigma^\mu\bar\lambda).$$

None of this depend on whether or not the superfield $V(x,\theta,\bar\theta)$ is taken to be in the Wess-Zumino gauge. We see that $f_{\mu\nu}(x)$, $\lambda(x)$ and $D(x)$ form a complete supersymmetry multiplet.

NOTE. The above transformation rules can be derived from the ones for the general scalar supermultiplet applied to $V(x,\theta,\bar\theta)$. Notice that the λ, $\bar\lambda$ and D components of V are redefined with respect to the form we used to write the general scalar supermultiplet.

At this point we could just guess a supersymmetric Lagrangian for $f_{\mu\nu}$, λ and D, however it is more convenient to find a superfield which contains these components and use it to build a Lagrangian.

The superfield which does the job is a <u>spinor superfield</u> W_α (with its conjugate $\bar W_{\dot\alpha} = (W_\alpha)^+$), which is defined as

$$W_\alpha = -\frac{1}{4}(\bar D\bar D) D_\alpha V(x,\theta,\bar\theta) ,$$

$$\bar W_{\dot\alpha} = -\frac{1}{4}(DD)\bar D_{\dot\alpha} V(x,\theta,\bar\theta) .$$

W_α is a chiral superfield as a consequence of $(\bar D)^3=0$, in the same way $\bar W_{\dot\alpha}$ is an antichiral superfield.

However W_α is not a general chiral spinor superfield, because W_α and $\bar W_{\dot\alpha}$ are related by an additional covariant constraint:

$$\underline{\bar D_{\dot\alpha} \bar W^{\dot\alpha} = D^\alpha W_\alpha .}$$

This constraint can be easily proven

$$\bar D_{\dot\alpha} \bar W^{\dot\alpha} = \varepsilon^{\dot\alpha\dot\beta} \bar D_{\dot\alpha} \bar W_{\dot\beta} = -\frac{1}{4} \varepsilon^{\dot\alpha\dot\beta} \bar D_{\dot\alpha} (DD) \bar D_{\dot\beta} V$$

$$= -\frac{1}{4} D^\alpha \bar D^2 D_\alpha V = D^\alpha W_\alpha .$$

An important property of W_α (and $\bar W_{\dot\alpha}$) is that it is invariant under gauge transformations. For Abelian gauge theories an extended gauge transformation takes the simple form

$$V(x,\theta,\bar\theta) \to V(x,\theta,\bar\theta) + \frac{i}{2}\left(\Lambda(x,\theta,\bar\theta) - \bar\Lambda(x,\theta,\bar\theta)\right)$$

with Λ a chiral superfield. We get

$$W_\alpha \to -\frac{1}{4} (\bar D \bar D) D_\alpha \left(V + \frac{i}{2}\Lambda - \frac{i}{2}\bar\Lambda\right)$$

$$= W_\alpha - \frac{i}{8} (\bar D \bar D) D_\alpha \Lambda \qquad (\text{since } D_\alpha \bar\Lambda = 0)$$

$$= W_\alpha + \frac{i}{8} \bar D^{\dot\beta} \{\bar D_{\dot\beta}, D_\alpha\} \Lambda \qquad (\text{since } \bar D_{\dot\beta} \Lambda = 0)$$

$$= W_\alpha .$$

Since W_α and $\bar W_{\dot\alpha}$ are both gauge invariant, there is no loss of generality in computing their components in the Wess-Zumino gauge:

$$\begin{cases} W_\alpha = -\frac{1}{4} (\bar D \bar D) D_\alpha V_{WZ}(x,\theta,\bar\theta) \\ \bar W_{\dot\alpha} = -\frac{1}{4} (DD) \bar D_{\dot\alpha} V_{WZ}(x,\theta,\bar\theta) \end{cases}$$

To compute the explicit expression for W_α it is more convenient to express the superfield in terms of y^μ, θ^α and $\bar\theta^{\dot\alpha}$:

$$V_{WZ} = \theta\sigma^\mu\bar\theta v_\mu(y) + i\theta\theta\,\bar\theta\bar\lambda(y) - i\bar\theta\bar\theta\,\theta\lambda(y) + \frac{1}{2}\theta\theta\,\bar\theta\bar\theta\left(D(y) - i\partial_\mu v^\mu(y)\right).$$

Then, by using $\sigma^\nu\bar\sigma^\mu - \eta^{\mu\nu} = 2\sigma^{\nu\mu}$, it is straightforward to find

$$D_\alpha V_{WZ} = (\sigma^\mu\bar\theta)_\alpha v_\mu(y) + 2i\theta_\alpha\,\bar\theta\bar\lambda(y) - i\bar\theta\bar\theta\,\lambda_\alpha(y) + \theta_\alpha\,\bar\theta\bar\theta D(y)$$

$$+ 2i(\sigma^{\mu\nu}\theta)_\alpha\,\bar\theta\bar\theta\,\partial_\mu v_\nu(y) + \theta\theta\,\bar\theta\bar\theta\left(\sigma^\mu\partial_\mu\bar\lambda(y)\right)_\alpha$$

and then, by using $\bar D\bar D\,\bar\theta\bar\theta = -4$:

$$\underline{W_\alpha = -i\lambda_\alpha(y) + \theta_\alpha D(y) + i(\sigma^{\mu\nu}\theta)_\alpha\left(\partial_\mu v_\nu(y) - \partial_\nu v_\mu(y)\right) + \theta\theta\left(\sigma^\mu\partial_\mu\bar\lambda(y)\right)_\alpha .}$$

We see that this chiral spinor superfield contains the component fields we found before by applying the susy transformations to $f_{\mu\nu}$. This supermultiplet is usually called "curl supermultiplet" or "field strength supermultiplet".

Since W_α is a chiral superfield $\int d^2\theta\, W^\alpha W_\alpha$ will be a susy invariant Lagrangian.
Explicitly

$$W^\alpha W_\alpha|_{\theta\theta} = -2i\lambda\sigma^\mu\partial_\mu\bar\lambda + D^2 - \frac{1}{2}(\sigma^{\mu\nu})^{\alpha\beta}(\sigma^{\rho\sigma})_{\alpha\beta} f_{\mu\nu} f_{\rho\sigma}.$$

By using

$$(\sigma^{\mu\nu})^{\alpha\beta}(\sigma^{\rho\sigma})_{\alpha\beta} = \frac{1}{2}(\eta^{\mu\rho}\eta^{\nu\sigma} - \eta^{\mu\sigma}\eta^{\nu\rho}) - \frac{i}{2}\varepsilon^{\mu\nu\rho\sigma}$$

we get

$$\int d^2\theta\, W^\alpha W_\alpha = -\frac{1}{2} f_{\mu\nu}f^{\mu\nu} - 2i\lambda\sigma^\mu\partial_\mu\bar\lambda + D^2 + \frac{i}{4}\varepsilon^{\mu\nu\rho\sigma} f_{\mu\nu} f_{\rho\sigma}.$$

Note that the first three terms are real, while the last one is purely imaginary.
The supersymmetric action for Abelian gauge fields is

$$\underline{\frac{1}{4}\left(\int d^2\theta\, W^\alpha W_\alpha + \int d^2\bar\theta\, \bar W_{\dot\alpha}\bar W^{\dot\alpha}\right) = -\frac{1}{4} f_{\mu\nu}f^{\mu\nu} - i\lambda\sigma^\mu\partial_\mu\bar\lambda + \frac{1}{2}D^2.}$$

Notice that the above action does not come from an F-term, it is indeed a D-term:

$$\int d^2\theta\, W^\alpha W_\alpha = \int d^2\theta\left(-\frac{1}{4}\bar D\bar D D^\alpha V\right)W_\alpha$$

$$= -\frac{1}{4}\int d^2\theta\, \bar D\bar D(D^\alpha V W_\alpha)$$

$$= \int d^2\theta\, d^2\bar\theta\, D^\alpha V W_\alpha.$$

Another interesting point of $W^\alpha W_\alpha|_{\theta\theta}$ is the presence of the $\varepsilon^{\mu\nu\rho\sigma} f_{\mu\nu} f_{\rho\sigma}$, it is a total derivative and we will discuss its role in the more general context of non-Abelian gauge theories. Notice that the above action shows that D is an auxiliary field.

· <u>The Fayet–Iliopoulos term</u>

In Abelian theories we can add also an extra term to the action. Under a gauge transformation

$$V \to V + \frac{i}{2}\Lambda - \frac{i}{2}\bar\Lambda$$

the D term of the V field is invariant, so we can add it to the action

$$\underline{\mathcal{L}_{FI} = \xi\int d^2\theta\, d^2\bar\theta\, V^A = \frac{1}{2}\xi D.}$$

where ξ is an arbitrary constant.
This is called the <u>Fayet–Iliopoulos term</u>, and it can be added to the action <u>only</u> for the <u>$U(1)$ subgroups</u> of the gauge group (we will see that for a <u>non-Abelian</u> group the D term is <u>not</u> gauge invariant).

· Gauge-invariant action for general gauge superfields

We now want to consider the case of a general gauge symmetry. To construct the gauge superfield Lagrangian we need, first of all, to find the non-Abelian generalization of the field strength superfield W_α.

We notice that the Abelian definition of W_α can be rewritten as

$$\begin{cases} W_\alpha = -\frac{1}{8}(\bar{D}\bar{D})\, e^{-2V} D_\alpha e^{2V} \\ \bar{W}_{\dot\alpha} = +\frac{1}{8}(DD)\, e^{2V} \bar{D}_{\dot\alpha} e^{-2V} \end{cases}.$$

These definitions can be adopted also in the general non-Abelian case. Let us compute how W_α and $\bar{W}_{\dot\alpha}$ transform under a gauge transformation. First we notice that

$$e^{-2V} D_\alpha e^{2V} \to e^{-i\lambda} e^{-2V}(D_\alpha e^{2V})\, e^{i\lambda} + e^{-i\lambda} D_\alpha e^{i\lambda}.$$

Using the fact that $\bar{D}_{\dot\alpha}$ commutes with λ, we get

$$W_\alpha \to e^{-i\lambda} W_\alpha e^{i\lambda} - \frac{1}{8} e^{-i\lambda}(\bar{D}\bar{D}) D_\alpha e^{i\lambda},$$

the second term vanishes, so we obtain that W_α and $\bar{W}_{\dot\alpha}$ transform covariantly in the non-Abelian case:

$$\begin{cases} W_\alpha \to e^{-i\lambda} W_\alpha e^{i\lambda} \\ \bar{W}_{\dot\alpha} \to e^{-i\bar\lambda} \bar{W}_{\dot\alpha} e^{i\bar\lambda} \end{cases}.$$

Again we can explicitly compute W_α in the Wess-Zumino gauge:

$$W_\alpha = -\frac{1}{8}(\bar{D}\bar{D})\, e^{-2V_{WZ}} D_\alpha e^{2V_{WZ}}$$

$$= -\frac{1}{4}(\bar{D}\bar{D}) D_\alpha V_{WZ} + \frac{1}{2}(\bar{D}\bar{D}) V_{WZ} D_\alpha V_{WZ} - \frac{1}{4}(\bar{D}\bar{D}) D_\alpha V_{WZ}^2$$

By using the expression of V_{WZ} in the coordinates $y^\mu, \theta^\alpha, \bar\theta^{\dot\alpha}$ we get

$$W_\alpha = -i\lambda_\alpha(y) + \theta_\alpha D(y) + i\sigma^{\mu\nu}{}_\alpha{}^\beta \theta_\beta F_{\mu\nu}(y) + (\theta\theta)\sigma^\mu_{\alpha\dot\beta} D_\mu \bar\lambda^{\dot\beta}(y)$$

where

$$F_{\mu\nu} \equiv \partial_\mu v_\nu - \partial_\nu v_\mu - i[v_\mu, v_\nu],$$

$$D_\mu \bar\lambda^{\dot\beta} \equiv \partial_\mu \bar\lambda^{\dot\beta} - i[v_\mu, \bar\lambda^{\dot\beta}],$$

that is $F_{\mu\nu}$ is the Yang-Mills field strength, while D_μ is the Yang-Mills gauge covariant derivative.

A gauge invariant supersymmetric action can be obtained from

$$\frac{1}{2}\int d^2\theta\, \mathrm{tr}\, W^\alpha W_\alpha = \mathrm{tr}\left[-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{i}{4}F_{\mu\nu}F_{\rho\sigma}\varepsilon^{\mu\nu\rho\sigma} - i\lambda\sigma^\mu D_\mu\bar\lambda + \frac{1}{2}D^2\right].$$

Notice that the above term is not real and it has no coupling constant.
To remedy these problems we introduce a complex gauge coupling τ:

$$\tau = \frac{\theta_{YM}}{2\pi} + \frac{4\pi i}{g^2}$$

where g is the gauge coupling constant and θ_{YM} is the Yang-Mills theta parameter.
The $N=1$ Yang-Mills action we want is

$$\frac{1}{8\pi} \operatorname{Im}\left[\tau \int d^4x \int d^2\theta \operatorname{tr} W^\alpha W_\alpha\right]$$

$$= \frac{1}{g^2}\int d^4x \operatorname{tr}\left[-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - i\lambda\sigma^\mu D_\mu\bar\lambda + \frac{1}{2}D^2\right]$$

$$- \frac{\theta_{YM}}{32\pi^2}\int d^4x \operatorname{tr} \varepsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma} .$$

NOTE. We could have written this action as the real part of $\int d^2\theta\, W^\alpha W_\alpha$ just by multiplying τ by a factor of i. We chose the above convention to follow the literature.

The $\varepsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}$ term is a total derivative, so it does not contribute to the action unless some topologically non trivial instantonic solution exist. In Abelian theories such solutions are not there, so the integral of this term on the whole space always vanishes. For non-Abelian theories one can have instanton solution and in this case the term can have a non-zero integral over spacetime, this integral is related to the topological properties of the instanton solution, namely to its winding number.
Given that the θ_{YM} term is related to topological properties and is a total derivative, it has only non-perturbative effects and has no role in perturbation theory.

· Renormalizable gauge theories with chiral superfields

Now we have all the ingredients we need to construct the general renormalizable gauge theory with chiral superfields. It is given by

$$\mathcal{L} = \mathcal{L}_{gauge} + \mathcal{L}_{matter} + \mathcal{L}_{FI}$$

$$= \frac{1}{8\pi}\operatorname{Im}\left[\tau\int d^2\theta \operatorname{tr} W^\alpha W_\alpha\right] + 2\sum_{\text{Abelian}} \xi^a \int d^2\theta\, d^2\bar\theta\, V^a$$

$$+ \int d^2\theta\, d^2\bar\theta\, \bar\Phi e^{2V}\Phi + \int d^2\theta\, W(\Phi) + \int d^2\bar\theta\, W^+(\bar\Phi)$$

$$= \frac{1}{g^2}\operatorname{Tr}\left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - i\lambda\sigma^\mu D_\mu\bar\lambda + \frac{1}{4}D^2\right) - \frac{\theta_{YM}}{32\pi^2}\operatorname{tr}\varepsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}$$

$$+ \sum_{\text{Abelian}}\xi^a D^a + (D_\mu z)^+ D^\mu z - i\bar\psi\bar\sigma^\mu D_\mu\psi + F^+F + \sqrt{2}\, i z^+\lambda\psi - \sqrt{2}\, i\,\bar\psi\bar\lambda z$$

$$+ z^+ D z - \left(\frac{\partial W}{\partial z^i}F^i + h.c.\right) - \left(\frac{1}{2}\frac{\partial^2 W}{\partial z^i\partial z^j}\psi^i\psi^j + h.c.\right) + \text{total derivative.}$$

As we did in the case without gauge symmetries, we can eliminate the auxiliary fields D and F from the Lagrangian by using their equations of motion:

$$F_i^+ = \frac{\partial W}{\partial z^i} \ ,$$

and

$$D^a = -z^+ T^a z - \xi^a \ ,$$

where it is understood that $\xi^a = 0$ if a does not take values in an Abelian factor of the gauge group. By substituting back into the Lagrangian one finds

$$\mathcal{L} = \frac{1}{g^2} \mathrm{Tr}\left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - i \lambda \sigma^\mu D_\mu \bar{\lambda}\right) - \frac{\theta_{YM}}{32\pi^2} \mathrm{tr}\, \varepsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma}$$

$$+ (D_\mu z)^+ D^\mu z - i \bar{\psi} \bar{\sigma}^\mu D_\mu \psi + i\sqrt{2}\, z^+ \lambda \psi - i\sqrt{2}\, \bar{\psi} \bar{\lambda} z$$

$$- \frac{1}{2} \frac{\partial^2 W}{\partial z^i \partial z^j} \psi^i \psi^j - \frac{1}{2} \left(\frac{\partial^2 W}{\partial z^i \partial z^j}\right)^+ \bar{\psi}^i \bar{\psi}^j - V(z^+, z) + \text{total derivative},$$

where the scalar potential $V(z^+, z)$ is now given by

$$\underline{V(z^+, z) = F^+ F + \frac{1}{2} D^2 = \sum_i \left|\frac{\partial W}{\partial z^i}\right|^2 + \frac{1}{2} \sum_a \left|z^+ T^a z + \xi^a\right|^2 .}$$

In this case the potential is the sum of two contributions: one coming from the F auxiliary field and one from the D auxiliary field. Notice that the two contributions are both non-negative, so the positivity of energy is respected.

· <u>General gauge theories with chiral superfields</u>

If we do not require renormalizability we can write the generalization of the σ-model we described for chiral superfields adapted to a theory with gauge invariance. The most generic matter Lagrangian gauge invariant under a group G is

$$\mathcal{L}_{matter} = \int d^2\theta\, d^2\bar{\theta}\; K(\bar{\Phi} e^{2V}, \Phi) + \int d^2\theta\, W(\Phi) + \int d^2\bar{\theta}\, W^+(\bar{\Phi}) ,$$

where $K(\bar{\Phi}_i, \Phi_i)$ is a real (globally) G-invariant function, and $W(\Phi_i)$ is a G-invariant function of the Φ_i.

The gauge Lagrangian is usually generalized as

$$\mathcal{L}_{gauge} = \frac{1}{16 g^2} \int d^2\theta\, h_{ab}(\Phi^i)\, W^{a\alpha} W^b{}_\alpha + h.c. ,$$

where $h_{ab} = h_{ba}$ transforms under G as the symmetric product of the adjoint representation with itself. To get back the standard renormalizable gauge Lagrangian we only need to take $\frac{1}{g^2} h_{ab} = \frac{\tau}{\pi i} \mathrm{tr}\, T^a T^b$.

The components expansion of the Lagrangian can be found in Bilal's review (section 7.2) or in Weinberg III (section 27.4).