VII. SUPERSYMMETRIC ACTIONS

In this section we will discuss how one can construct supersymmetric theories by using the superfield formalism. As a first step we will consider the analogous of the Wess-Zumino model, that is we will focus on theories which only contain chiral multiplets. The next step, which we will discuss in another section, will be the introduction of gauge symmetries in supersymmetric theories.

To build susy theories with chiral multiplets using the superspace formalism we need, of course, to identify the superfield which corresponds to the chiral multiplet. This superfield will be constructed by applying susy invariant constraints to the general scalar $N=1$ superfield.

• N=1 chiral superfields

A chiral superfield can be obtained by imposing the constraints given on page 6.6 on the superfield components. A more elegant way, equivalent to the previous constraints, is to impose on a superfield the covariant constraint

$$\bar{D}_{\dot\alpha}\, \Phi = 0 .$$

This constraint is manifestly susy invariant given that

$$\{ \bar{D}_{\dot\alpha}, Q_\beta \} = \{ \bar{D}_{\dot\alpha}, \bar{Q}_{\dot\beta} \} = 0 .$$

We must now find the most general solution to the covariant constraint. We can simplify our task by defining new bosonic coordinates on the superspace:

$$y^\mu \equiv x^\mu + i\theta\sigma^\mu\bar\theta \quad , \quad y^{\mu +} = x^\mu - i\theta\sigma^\mu\bar\theta .$$

Since

$$\bar{D}_{\dot\alpha}\, y^\mu = 0 \quad ,$$
$$\bar{D}_{\dot\alpha}\, \theta^\alpha = 0 \quad ,$$

any function of y^μ and θ^α (but not $\bar\theta_{\dot\alpha}$) satisfies the covariant constraint

$$\bar{D}_{\dot\alpha}\, \Phi(y,\theta) = 0 .$$

Furthermore, this is actually the most general solution, given that $\bar{D}_{\dot\alpha}$ satisfies the chain rule.

Thus we may write the most general $N=1$ chiral superfield as

$$\Phi(y,\theta) = z(y) + \sqrt{2}\,\theta\psi(y) - \theta\theta F(y),$$

where $z(y)$, $F(y)$ are complex scalar fields, while $\psi^\alpha(y)$ is a complex left-handed Weyl spinor (the $\sqrt{2}$ in front of $\theta\psi$ is a convention).

We immediately see that the field content of this superfield is exactly the same as the one we found before with the direct construction method: we have 4 real bosonic and 4 real fermionic degrees of freedom off-shell; this is twice the number in the on-shell fundamental $N=1$ massive representation.

As we will see F is an auxiliary field, the on-shell degrees of freedom are given by a complex scalar and a Weyl fermion $(2+2=4$ degrees of freedom$)$.

We can obtain the full $\theta, \bar\theta$ component expansion for the chiral superfield (by using the Fierz identities):

$$\Phi(y,\theta) = z(x) + \sqrt{2}\,\theta\psi(x) - \theta\theta F(x)$$
$$+ i\,\theta\sigma^\mu\bar\theta\,\partial_\mu z(x) - \frac{i}{\sqrt{2}}(\theta\theta)\,\partial_\mu\psi(x)\sigma^\mu\bar\theta - \frac{1}{4}(\theta\theta)(\bar\theta\bar\theta)\,\Box z(x).$$

An infinitesimal $N=1$ susy transformation on the chiral superfield yields

$$\begin{cases} \delta z = \sqrt{2}\,\varepsilon\psi \\ \delta\psi = -\sqrt{2}\,\varepsilon F + \sqrt{2}\,i\,\sigma^\mu\bar\varepsilon\,\partial_\mu z \\ \delta F = +\sqrt{2}\,i\,\partial_\mu\psi\,\sigma^\mu\bar\varepsilon \end{cases}.$$

Notice that <u>$\delta F(x)$ is a total derivative</u>.

. We can also define <u>antichiral</u> superfields by imposing the condition

$$D_\alpha\bar\Phi = 0.$$

It is easy to see that if Φ is a chiral superfield, then Φ^+ is an antichiral superfield.

An important property of chiral superfields is that the product of chiral superfields is still chiral

$$\bar D_{\dot\alpha}\Phi^i = 0 \quad\Rightarrow\quad \bar D_{\dot\alpha}\Big(\prod_i \Phi^{i}\Big) = 0,$$

and analogously for antichiral superfields. However the sum or the product of chiral <u>and</u> antichiral superfields is <u>not</u> chiral (nor antichiral).

To derive the above relations it is useful to rewrite the susy generators and the derivatives in the $y^\mu, \theta, \bar\theta$ coordinates:

$$Q_\alpha = -i\frac{\partial}{\partial\theta^\alpha}\ ,\quad \bar Q_{\dot\alpha} = i\frac{\partial}{\partial\bar\theta^{\dot\alpha}} + 2\theta^\beta\sigma^\mu_{\beta\dot\alpha}\frac{\partial}{\partial y^\mu}\ ,$$

$$D_\alpha = \frac{\partial}{\partial\theta^\alpha} + 2i\sigma^\mu_{\alpha\dot\beta}\bar\theta^{\dot\beta}\frac{\partial}{\partial y^\mu}\ ,\quad \bar D_{\dot\alpha} = \frac{\partial}{\partial\bar\theta^{\dot\alpha}}\ .$$

• N=1 globally supersymmetric actions

To build a supersymmetric action we must find some quantities which are invariant under supersymmetry or whose variation is a spacetime derivative.

The only completely invariant quantities are spacetime constants (given that $\{Q_\alpha, \bar Q_{\dot\alpha}\} \propto P_\mu$), so the only possibility is to construct quantities whose variation is a total derivative.

When we analyzed the general N=1 superfield and the chiral superfield we already found two candidates:

- the D-term (the $(\theta\theta)(\bar\theta\bar\theta)$ term) of a general superfield;
- the F-term (the $(\theta\theta)$ term) of a chiral superfield (and analogously the $\bar\theta\bar\theta$ term for the antichiral case).

These two quantities are the building blocks to construct susy invariant theories.

Of course, we can take the D-term of a generic superfield, including possible products of superfields, and analogously for the F-term which can also come from a product of chiral superfields.

The most general supersymmetric Lagrangian for a set of chiral superfields Φ is

$$\mathcal{L} = [K(\Phi,\bar\Phi)]_D + [W(\Phi)]_F + [W(\Phi)]_F^+$$
$$= \int d^2\theta\, d^2\bar\theta\ K(\Phi,\bar\Phi) + \int d^2\theta\ W(\Phi) + \int d^2\bar\theta\ W^+(\bar\Phi)\ ,$$

where $K(\Phi,\bar\Phi)$ is a real scalar function of the chiral Φ and antichiral $\bar\Phi$ superfields (K is also known as Kahler Potential), and $W(\Phi)$ is an holomorphic function of the chiral fields only. When W is expressed as a function only of elementary chiral superfields (and not their superderivatives or spacetime derivatives) it is known as the superpotential.

Notice that we can always express an integral over $d^2\theta\, d^2\bar\theta$ as an integral on only $d^2\theta$:

$$\int d^2\theta\, d^2\bar\theta\ F = -\frac{1}{4}\int d^2\theta\ \bar D^2 F ,$$

this is related to the fact that $\bar D^2 F$ is a chiral superfield (exercise).
This means that a D-term can also be expressed as an F-term, but the contrary is in general not possible. Thus we usually call F-terms only the ones which can not be expressed as an integral over the whole superspace, while the others are called D-terms.

· The Wess-Zumino model

To explore the world of susy theories and understand the role of the Kahler potential K and of the superpotential W, we will consider the simple case with just one chiral superfield Φ.

The most general renormalizable susy Lagrangian is

$$\mathcal{L} = \int d^2\theta\, d^2\bar\theta\ \bar\Phi\Phi - \left[\int d^2\theta \left(\frac{1}{2} m \Phi^2 + \frac{1}{3} g \Phi^3\right) + h.c.\right].$$

Let's consider the first term

$$\int d^2\theta\, d^2\bar\theta\ \bar\Phi\Phi .$$

One can compute the D-term of the product $\bar\Phi\Phi$:

$$\underline{\bar\Phi\Phi\big|_{\theta\theta\bar\theta\bar\theta}} = -\frac{1}{4}(\Box z^+) z - \frac{1}{4} z^+ \Box z + F^+F + \frac{1}{2}\partial^\mu z^+ \partial_\mu z$$

$$+\frac{i}{2}\partial_\mu\psi\sigma^\mu\bar\psi - \frac{i}{2}\psi\sigma^\mu\partial_\mu\bar\psi$$

$$= \underline{\partial_\mu z^+ \partial^\mu z + \frac{i}{2}\left(\partial_\mu\psi\sigma^\mu\bar\psi - \psi\sigma^\mu\partial_\mu\bar\psi\right) + F^+F + \text{total derivative}.}$$

This shows that the Kahler potential contains the kinetic terms for the component fields (apart from a term containing the auxiliary field F).

Any term coming from the Kahler potential contains derivatives, so it can only give kinetic terms (for the physical fields) but not interaction terms. To get the interactions we must consider the superpotential.

Instead of just studying the given superpotential, it is more interesting to treat the general case and then adapt it to the case at hand.
Using the y^μ and θ variables it is easy to find the expansion

$$W(\Phi) = W(z(y)) + \sqrt{2}\,\frac{\partial W}{\partial z}\,\theta\psi(y)$$
$$-\theta\theta\left(\frac{\partial W}{\partial z}F(y) + \frac{1}{2}\frac{\partial^2 W}{\partial z\partial z}\psi(y)\psi(y)\right)$$

where $\frac{\partial W}{\partial z}$ and $\frac{\partial^2 W}{\partial z \partial z}$ are evaluated at $z(y)$.

The complete action reads

$$S = \int d^4x\left[|\partial_\mu z|^2 - i\psi\sigma^\mu\partial_\mu\bar{\psi} + F^+F - \frac{\partial W}{\partial z}F + h.c. - \frac{1}{2}\frac{\partial^2 W}{\partial z\partial z}\psi\psi + h.c.\right].$$

The F field is clearly an auxiliary field (it lacks a kinetic term), so we can integrate it out by solving its equations of motion:

$$F^+ = \frac{\partial W}{\partial z}.$$

By substituting into the action we get

$$S = \int d^4x\left[|\partial_\mu z|^2 - i\psi\sigma^\mu\partial_\mu\bar{\psi} - \left|\frac{\partial W}{\partial z}\right|^2 - \frac{1}{2}\frac{\partial^2 W}{\partial z\partial z}\psi\psi - \frac{1}{2}\left(\frac{\partial^2 W}{\partial z\partial z}\right)^+\bar{\psi}\bar{\psi}\right].$$

We can notice that the scalar potential V is determined by the superpotential W:

$$V = \left|\frac{\partial W}{\partial z}\right|^2.$$

We notice moreover that the potential is always <u>non-negative</u>, in agreement with the property that the energy in a susy theory must be positive or zero.

It is now straightforward to derive the explicit form of the action in the example we considered before with

$$W(\Phi) = \frac{m}{2}\Phi^2 + \frac{g}{3}\Phi^3.$$

In this case

$$\frac{\partial W}{\partial z} = mz + gz^2, \quad \frac{\partial^2 W}{\partial z^2} = 2gz + m,$$

so we get the following action, which is the interesting Wess-Zumino action:

$$S_{WZ} = \int d^4x\left[|\partial_\mu z|^2 - i\psi\sigma^\mu\partial_\mu\bar{\psi} - m^2|z|^2 - \frac{m}{2}(\psi\psi + \bar{\psi}\bar{\psi})\right.$$
$$\left. - mg\left(z^+z^2 + (z^+)^2 z\right) - g^2|z|^4 + g\left(z\psi\psi + z^+\bar{\psi}\bar{\psi}\right)\right].$$

This action describes a complex scalar and a fermion both of mass m. Note that the Yukawa interaction appear with a coupling g that is related by susy to the bosonic couplings mg and g^2.

We can easily check explicitly that the Kahler term is susy invariant:

$$\delta_{\varepsilon,\bar\varepsilon} K = i(\varepsilon Q + \bar\varepsilon \bar Q) K$$

$$= \frac{\partial}{\partial\theta^\alpha}(-\varepsilon^\alpha K) + \frac{\partial}{\partial\bar\theta^{\dot\alpha}}(-\bar\varepsilon^{\dot\alpha} K) + \partial_\mu\left[-i(\varepsilon\sigma^\mu\bar\theta - \theta\sigma^\mu\bar\varepsilon)K\right]$$

when we integrate over the whole superspace the first two terms give zero (they contain at most three θ's) and the last term is a total derivative.

Let now discuss a few relevant points of the model.

- The superpotential can be chosen not to contain superderivatives or spacetime derivatives, and this is its standard form. A piece with derivatives / superderivatives is either not chiral or can be rewritten as a D-term. For example if we act with D_α on a chiral field we get something which is non-chiral, on the other hand if we include a term like

$$\Phi \cdot \bar D \bar D S$$

where Φ is chiral we can always rewrite it as

$$\bar D \bar D (\Phi S)$$

which can be expressed as a D-term.

- The Kahler potential is defined up to chiral superfield. A chiral superfield has no D-term which can contribute to the action, so

$$K \to K + \Lambda(\Phi) + \Lambda^+(\bar\Phi) \equiv K'$$

gives the same action.

- From the compact expansion of chiral superfields we can see that the Kahler term, written as a function of Φ and $\bar\Phi$ but not their derivatives, contains terms with at most two spacetime derivatives

$$[K]_D = \cdots g(\phi,\bar\phi)\,\partial_\mu\bar\phi\,\partial^\mu\phi + \cdots$$

- <u>Renormalizability conditions</u>

The Lagrangian density in a renormalizable theory can only contain operators with dimensionality (counting powers of energy) four or less. From the relation

$$\{Q_\alpha, \bar Q_{\dot\alpha}\} = -2i\sigma^\mu_{\alpha\dot\beta}\partial_\mu$$

we get

$$\left[\frac{\partial}{\partial\theta^\alpha}\right] = \left[\frac{\partial}{\partial\bar\theta^{\dot\alpha}}\right] = 1/2 \,,$$

so that

$$[\theta^\alpha] = [\bar\theta^{\dot\alpha}] = -1/2 .$$

The F- and D-terms in a superfield contain two and four θ powers respectively, so, if $d(\Phi)$ is the dimension of the superfield Φ, the dimension of its F- and D-terms is

$$[\Phi_D] = d(\Phi) + 2 ,$$
$$[\Phi_F] = d(\Phi) + 1 .$$

Thus in a renormalizable field theory W and K consist of operators with dimensionality at most three and two, respectively:

$$[K] \leq 2 ,$$
$$[W] \leq 3 .$$

The dimensionality of an elementary scalar superfield is that of an elementary scalar field $[\Phi] = 1$, so the superpotential W can contain at most three factors of Φ, without superderivatives or spacetime derivatives. (Terms with superderivatives in W can be expressed as D-terms; terms with spacetime derivatives in renormalizable theories would necessarily be of the form $\partial_\mu \partial^\mu \Phi$, as a consequence of Lorentz invariance, so they do not contribute to the action).

The same analysis shows that K is at most a quadratic function of Φ and $\bar{\Phi}$, without derivatives. But any term which contains just Φ or just $\bar{\Phi}$ would be chiral, so it would not contribute to the action with a D-term, so $[K(\Phi, \bar{\Phi})]_D$ receives contributions only from terms that involve both Φ and $\bar{\Phi}$, namely

$$K(\Phi, \bar{\Phi}) = \bar{\Phi}\Phi .$$

. This analysis shows that the interacting Wess-Zumino model we described before is the most general renormalizable susy Lagrangian of one chiral superfield.

. Generalization to an arbitrary set of chiral superfields

The above construction and results can be easily generalized to the case of a set of chiral superfields Φ_i, $i = 1, \dots, n$.

The general form of the Lagrangian is

$$\mathcal{L} = \int d^2\theta \, d^2\bar{\theta} \, K(\Phi_i, \bar{\Phi}_i) + \int d^2\theta \, W(\Phi_i) + \int d^2\bar{\theta} \, W^+(\bar{\Phi}_i) .$$

where again K is a real function of $\{\Phi_i\}$ and $\{\bar{\Phi}_i\}$ and W is an holomorphic function of $\{\Phi_i\}$.

In the renormalizable case we have

$$K = K^i{}_j \, \Phi_i \, \bar{\Phi}^j ,$$

which, by a redefinition of the fields (notice that $K^i{}_j$ must be positive definite to have a well-defined action) can be brought to the standard form

$$K = \sum_i \bar{\Phi}^i \Phi_i .$$

For the superpotential W we have

$$W(\Phi_i) = \sum_i a_i \Phi_i + \sum_{i,j} m_{ij} \Phi_i \Phi_j + \sum_{i,j,k} g_{ijk} \Phi_i \Phi_j \Phi_k ,$$

where m_{ij} and g_{ijk} are totally symmetric in the indices (because the Φ_i fields commute among themselfs).

The action is given by

$$S = \int d^4x \left[|\partial_\mu z_i|^2 - i\psi_i \sigma^\mu \partial_\mu \bar{\psi}_i + F_i^+ F_i - \frac{\partial W}{\partial z_i} F_i + h.c. - \frac{1}{2} \frac{\partial^2 W}{\partial z_i \partial z_j} \psi_i \psi_j + h.c. \right].$$

we can again solve for the auxiliary fields:

$$S = \int d^4x \left[|\partial_\mu z_i|^2 - i\psi_i \sigma^\mu \partial_\mu \bar{\psi}_i - \left|\frac{\partial W}{\partial z_i}\right|^2 - \frac{1}{2} \frac{\partial^2 W}{\partial z_i \partial z_j} \psi_i \psi_j - \frac{1}{2} \left(\frac{\partial^2 W}{\partial z_i \partial z_j}\right)^+ \bar{\psi}_i \bar{\psi}_j \right],$$

and the potential is

$$V = \sum_i F_i^+ F_i = \sum_i \left| \frac{\partial W}{\partial z_i} \right|^2 .$$

NOTE. In the case of a theory of one chiral superfield the linear term $a\Phi$ in the superpotential can always be eliminated by a field redefinition provided that m and/or g are non-vanishing. If $m = g = 0$ the $a\Phi$ term gives just a contribution to the auxiliary field lagrangian and is irrelevant. (exercise)

- R-charge

There is a symmetry which is often important in supersymmetric theories: the R-symmetry. We already saw that the N=1 susy algebra is invariant under a U(1) R-symmetry under which

$$\begin{cases} [Q_\alpha, R] = Q_\alpha \\ [\bar{Q}_{\dot\alpha}, R] = -\bar{Q}_{\dot\alpha} \end{cases}$$

From the representation of Q_α and $\bar{Q}_{\dot\alpha}$ on the superspace we see that θ has R-charge -1, while $\bar\theta$ has R-charge $+1$. This implies that, under a finite R-symmetry transformation chiral fields transform as

$$R\,\Phi(\theta, y) = e^{ir\alpha} \Phi(e^{-i\alpha}\theta, y) ,$$

$$R\,\bar{\Phi}(\bar\theta, y^+) = e^{-ir\alpha} \bar{\Phi}(e^{i\alpha}\bar\theta, y^+) ,$$

where r is the R-charge of the superfield. For the component fields we get

$$\begin{aligned} R:\ z &\to e^{ir\alpha} z &&: \quad R(z) = r , \\ \psi &\to e^{i(r-1)\alpha} \psi &&: \quad R(\psi) = r-1 , \\ F &\to e^{i(r-2)\alpha} F &&: \quad R(F) = r-2 . \end{aligned}$$

For the superpotential $W(\Phi)$ to respect R-symmetry its F-term must have vanishing R-charge, that is

$$R(W(\Phi)) = +2 .$$

On the other hand the D-term of a superfield has the same R-charge as the superfield itself, so for the Kahler potential to respect R-symmetry it must have R-charge zero:

$$R(K(\Phi, \bar{\Phi})) = 0 .$$

For a single chiral field this is achieved if all the terms in the Kahler potential contain an equal number of Φ and $\bar{\Phi}$ terms.

Notice that in general it is not necessary to impose R-symmetry in a supersymmetric theory, or, even if it is present, it could be spontaneously broken.

· General Kahler potentials (see Bilal's and Lykken's reviews).

In some cases it is useful to consider non-renormalizable Lagrangians of the form

$$\mathcal{L} = \int d^2\theta\, d^2\bar{\theta}\ K[\Phi_i, \bar{\Phi}_j] + \int d^2\theta\ W(\Phi_i) + \int d^2\bar{\theta}\ W^+(\bar{\Phi}_i)$$

where $W(\Phi)$ and $K[\Phi, \bar{\Phi}]$ are arbitrary functions of Φ and $\bar{\Phi}$ but not their derivatives. We already discussed the Lagrangian which comes from a generic superpotential, so now we will concentrate on the Kahler potential.

In general, the D-term of an arbitrary Kahler potential can be written in the form

$$\int d^2\theta\, d^2\bar{\theta}\ K(\Phi, \bar{\Phi}) = K_i{}^j \left(\partial_\mu z^i \partial^\mu \bar{z}_j - \frac{i}{2} \psi^i \sigma^\mu \partial_\mu \bar{\psi}_j + \frac{i}{2} \partial_\mu \psi^i \sigma^\mu \bar{\psi}_j + F^i F^+_j \right)$$
$$+ \frac{i}{4} K_{ij}{}^k \left(\psi^i \sigma^\mu \bar{\psi}_k \partial_\mu z^j + \psi^j \sigma^\mu \bar{\psi}_k \partial_\mu z^i - 2i \psi^i \psi^j \bar{F}_k \right) + h.c.$$
$$+ \frac{1}{4} K_{ij}{}^{kl} \psi^i \psi^j \bar{\psi}_k \bar{\psi}_l + \text{total derivative},$$

where we defined

$$K_i{}^j = \frac{\partial^2}{\partial z^i \partial \bar{z}_j} K(\Phi, \bar{\Phi}) .$$

$K_i{}^j$ can be interpreted as the metric of a complex Riemannian manifold, called Kahler manifold. $K_i{}^j$ is called the Kahler metric. Notice that, since K is real, $K_i{}^j$ is hermitian, moreover, in order to have a consistent theory we must also require it to be positive definite and non-singular. The interpretation of $K_i{}^j$ as a metric of a complex manifold implies that models with complicated Lagrangians can be characterized by the algebraic geometry of the Kahler manifold. Model of this kind are a susy version of ordinary σ-models, and are useful to describe the Goldstones coming from a spontaneously broken global symmetry.