II. THE POINCARÉ GROUP

The Poincaré group contains the Lorentz group and the translations.

· The Lorentz group

The Lorentz group is defined as the group of transformations which leave the Minkowski metric invariant. In this course we will adopt the following convention for the metric

$$\eta_{\mu\nu} = (+1,-1,-1,-1).$$

The Lorentz group, also denoted as $SO(3,1)$, is a non-compact group with 6 generators:

$$- \text{3 rotations} : J_i \qquad i=1,2,3$$
$$- \text{3 boosts} : K_i \qquad i=1,2,3$$

Their commutation relations are given by

$$\begin{cases} [J_i, J_j] = i\varepsilon_{ijk} J_k \\ [K_i, K_j] = -i\varepsilon_{ijk} J_k \\ [J_i, K_j] = i\varepsilon_{ijk} K_k \end{cases}.$$

In order to study the Lorentz group representations it is useful to introduce the linear combinations

$$\begin{cases} S_i \equiv \frac{1}{2}(J_i + iK_i) \\ T_i \equiv \frac{1}{2}(J_i - iK_i) \end{cases},$$

these operators satisfy the following relations

$$\begin{cases} [S_i, S_j] = i\varepsilon_{ijk} S_k \\ [T_i, T_j] = i\varepsilon_{ijk} T_k \end{cases}, \qquad [S_i, T_j] = 0 .$$

However S_i and T_i are not Hermitian:

$$S_i^+ = T_i \quad ,$$

so they define a complexified version of the $SU(2)\times SU(2)$ group.

Nevertheless we can classify the Lorentz group representations in terms of the representations of $SU(2)\times SU(2)$ complexified.

This group is also equivalent to $SL(2,\mathbb{C})$, which is the universal cover of $SO(3,1)$, and is given by the 2×2 complex matrices of unit determinant.

To prove the equivalence we can just rewrite the spacetime coordinates x^μ by using a matrix notation:

$$x^\mu \to x^\mu \sigma_\mu$$

where $\sigma_0 = \mathbb{1}_{2\times 2}$ and σ_i are the Pauli matrices. The determinant of $x^\mu\sigma_\mu$ gives the norm of the x^μ vector:

$$\det(x^\mu \sigma_\mu) = x^\mu x_\mu .$$

The Lorentz transformations leave $x^\mu x_\mu$ invariant, so they are given by the transformations which leave $\det(x^\mu\sigma_\mu)$ unchanged, or explicitly by

$$x^\mu\sigma_\mu \to A\, x^\mu \sigma_\mu\, A^+$$

where A is a complex 2×2 matrix of unit determinant (up to an irrelevant phase), that is an element of $SL(2,\mathbb{C})$.

· <u>The Poincaré group</u>

If we add translations to the Lorentz transformations we get the Poincaré group. The spacetime translations generators are denoted by P_μ and they satisfy the additional commutation relations

$$[P_\mu, P_\nu] = 0$$

$$[J_i, P_j] = i\,\varepsilon_{ijk} P_k \qquad [J_i, P_0] = 0$$

$$[K_i, P_j] = +i\,\delta_{ij} P_0 \qquad [K_i, P_0] = -i\, P_i$$

The Poincaré algebra can also be rewritten in a covariant form. We define the combinations $M_{\mu\nu}$ (which satisfy $M_{\mu\nu} = -M_{\nu\mu}$)

$$M_{0i} = K_i$$

$$M_{ij} = \varepsilon_{ijk} J_k .$$

The Poincaré algebra reads

$$\begin{cases} [P_\mu, P_\nu] = 0 \\ [M_{\mu\nu}, M_{\rho\sigma}] = i g_{\nu\rho} M_{\mu\sigma} - i g_{\mu\rho} M_{\nu\sigma} - i g_{\nu\sigma} M_{\mu\rho} + i g_{\mu\sigma} M_{\nu\rho} \\ [M_{\mu\nu}, P_\rho] = -i g_{\rho\mu} P_\nu + i g_{\rho\nu} P_\mu . \end{cases}$$

· <u>Casimirs</u> of the Poincaré group.

The Poincaré group has two Casimirs:

$$P^2 \equiv P_\mu P^\mu$$

and

$$W^2 \equiv W_\mu W^\mu$$

where W^μ is the Pauli-Lubanski vector:

$$W^\mu \equiv \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma} P_\nu M_{\rho\sigma} .$$

The irreducible representations have definite values of P^2 and W^2. $P^2 = m^2$ gives the mass of the particles, while W^2 gives the spin for massive particles and W^μ is related to the helicity in the massless case.

· Spinors

The representations of the Lorentz group can be labelled by the corresponding representations under $SU(2)\times SU(2)$, namely Lorentz group representations are identified by a pair

$$(a, b)$$

where a, b denote the representation under the first/second $SU(2)$ subgroup.

The fermions are given by the representations

$$\left(\tfrac{1}{2}, 0\right) \Rightarrow \text{"left-handed" Weyl fermion},$$

$$\left(0, \tfrac{1}{2}\right) \Rightarrow \text{"right-handed" Weyl fermion},$$

and have spin $1/2$.

There is also an equivalent way to define the fermions as the base representations of $Sl(2,\mathbb{C})$. A spinor is a two complex component object $\psi = \begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}$ which transforms under an element $M \in Sl(2,\mathbb{C})$ as

$$\psi_\alpha \rightarrow \psi'_\alpha = M_\alpha{}^\beta \psi_\beta .$$

The complex conjugate of this representation of $Sl(2,\mathbb{C})$ gives an independent representation not equivalent to the previous; we denote a two-component object $\bar{\psi}$ in this representation by a bar and a dotted index

$$\bar{\psi}_{\dot\alpha} \rightarrow \bar{\psi}'_{\dot\alpha} = M^{*}_{\dot\alpha}{}^{\dot\beta} \bar{\psi}_{\dot\beta} .$$

· It's easy to see that we can identify

$$\bar{\psi}_{\dot\alpha} \equiv (\psi_\alpha)^{*} .$$

· We can also check that ψ_α corresponds to the $(1/2, 0)$ representation and $\bar{\psi}_{\dot\alpha}$ to the $(0, 1/2)$. To prove this we rewrite M and M^* as

$$M = \exp\left(a_j \sigma_j + i b_j \sigma_j\right) \qquad a_j, b_j \in \mathbb{R}.$$

$$M^{*} = \exp\left(a_j \sigma_j - i b_j \sigma_j\right)$$

This shows that M and M^* correspond to the two complexified $SU(2)$ subgroup of the Lorentz group.

NOTE. In some other notations the dotted spinors are denoted by a $+$ and not a bar:

$$\bar{\psi}_{\dot\alpha} \rightarrow \psi^{+}_{\dot\alpha} .$$

IMPORTANT NOTE. Spinors are anticommuting objects, thus if we exchange two fermions in an expression we must change the sign. For example:

$$\psi_1 \chi_2 = -\chi_2 \psi_1 \; ; \; \psi_1 \bar{\chi}_{\dot 2} = -\chi_{\dot 2} \psi_1 \; ; \ldots$$

· Notations and conventions (follows Bilal's review).

We introduce the antisymmetric tensors $\varepsilon^{\alpha\beta}$ and $\varepsilon_{\alpha\beta}$

$$\varepsilon^{\alpha\beta} = \varepsilon^{\dot\alpha\dot\beta} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad , \quad \varepsilon_{\alpha\beta} = \varepsilon_{\dot\alpha\dot\beta} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

which are used to raise and lower indices:

$$\psi^\alpha = \varepsilon^{\alpha\beta}\psi_\beta \ , \quad \psi_\alpha = \varepsilon_{\alpha\beta}\psi^\beta \ , \quad \bar\psi^{\dot\alpha} = \varepsilon^{\dot\alpha\dot\beta}\bar\psi_{\dot\beta} \ , \quad \bar\psi_{\dot\alpha} = \varepsilon_{\dot\alpha\dot\beta}\bar\psi^{\dot\beta} ,$$

this is consistent since

$$\varepsilon_{\alpha\beta}\,\varepsilon^{\beta\gamma} = \varepsilon^{\gamma\beta}\varepsilon_{\beta\alpha} = \delta_\alpha^\gamma \quad \text{and} \quad \varepsilon_{\dot\alpha\dot\beta}\,\varepsilon^{\dot\beta\dot\gamma} = \varepsilon^{\dot\gamma\dot\beta}\varepsilon_{\dot\beta\dot\alpha} = \delta_{\dot\alpha}^{\dot\gamma} .$$

· As a convention repeated indices contracted like

$${}^{\alpha}{}_{\alpha} \qquad (\underline{\text{undotted}}: \text{from upper left to lower right})$$

$${}_{\dot\alpha}{}^{\dot\alpha} \qquad (\underline{\text{dotted}}: \text{from lower left to upper right})$$

can be suppressed (eg. $\psi^\alpha\chi_\alpha \equiv \psi\chi$, $\bar\psi_{\dot\alpha}\bar\chi^{\dot\alpha} \equiv \bar\psi\bar\chi$).

In particular we have

$$\psi\chi \equiv \psi^\alpha\chi_\alpha = \varepsilon^{\alpha\beta}\psi_\beta\chi_\alpha = -\varepsilon^{\alpha\beta}\psi_\alpha\chi_\beta = -\psi_\alpha\chi^\alpha = \chi^\alpha\psi_\alpha = \chi\psi ,$$

and analogously

$$\bar\psi\bar\chi = \bar\psi_{\dot\alpha}\bar\chi^{\dot\alpha} = \bar\chi\bar\psi .$$

Moreover we get

$$(\psi\chi)^+ = \bar\chi_{\dot\alpha}\bar\psi^{\dot\alpha} = \bar\chi\bar\psi = \bar\psi\bar\chi .$$

· The analogous of the γ^μ matrices is given by the σ^μ matrices:

$$(\sigma^\mu)_{\alpha\dot\alpha} = (\mathbb{1}, -\sigma_i)_{\alpha\dot\alpha} ,$$
$$(\bar\sigma^\mu)^{\dot\alpha\alpha} = \varepsilon^{\dot\alpha\dot\beta}\varepsilon^{\alpha\beta}(\sigma^\mu)_{\beta\dot\beta} = (\mathbb{1}, \sigma_i)^{\dot\alpha\alpha} .$$

These matrices have an undotted an a dotted index.

Combinations of spinors with the σ^μ matrices can be built:

$$\psi\sigma^\mu\bar\chi \equiv \psi^\alpha\sigma^\mu{}_{\alpha\dot\beta}\bar\chi^{\dot\beta} \quad , \quad \bar\psi\bar\sigma^\mu\chi \equiv \bar\psi_{\dot\alpha}\bar\sigma^{\mu\,\dot\alpha\beta}\chi_\beta .$$

One can check that

$$\bar\psi\bar\sigma^\mu\chi = -\chi\sigma^\mu\bar\psi = (\bar\chi\bar\sigma^\mu\psi)^* = -(\psi\sigma^\mu\bar\chi)^* .$$

Useful relations:

$$\psi\sigma^\mu\bar\sigma^\nu\chi = \chi\sigma^\nu\bar\sigma^\mu\psi = (\bar\chi\bar\sigma^\nu\sigma^\mu\bar\psi)^* = (\bar\psi\bar\sigma^\mu\sigma^\nu\bar\chi)^*$$

the Fierz rearrangement identity

$$\chi_\alpha(\xi\eta) = -\xi_\alpha(\eta\chi) - \eta_\alpha(\chi\xi) ,$$

and the reduction identities

$$\sigma^{\mu}_{\alpha\dot{\alpha}}\,\bar{\sigma}_{\mu}^{\dot{\beta}\beta} = +2\,\delta_{\alpha}^{\beta}\,\delta_{\dot{\alpha}}^{\dot{\beta}}$$

$$\sigma^{\mu}_{\alpha\dot{\alpha}}\,\sigma_{\mu\beta\dot{\beta}} = +2\,\varepsilon_{\alpha\beta}\,\varepsilon_{\dot{\alpha}\dot{\beta}}$$

$$\bar{\sigma}^{\mu\dot{\alpha}\alpha}\,\bar{\sigma}_{\mu}^{\dot{\beta}\beta} = +2\,\varepsilon^{\alpha\beta}\,\varepsilon^{\dot{\alpha}\dot{\beta}}$$

$$[\sigma^{\mu}\bar{\sigma}^{\nu}+\sigma^{\nu}\bar{\sigma}^{\mu}]_{\alpha}{}^{\beta} = +2\eta^{\mu\nu}\,\delta_{\alpha}^{\beta}$$

$$[\bar{\sigma}^{\mu}\sigma^{\nu}+\bar{\sigma}^{\nu}\sigma^{\mu}]^{\dot{\beta}}{}_{\dot{\alpha}} = +2\eta^{\mu\nu}\,\delta_{\dot{\alpha}}^{\dot{\beta}}$$

$$\bar{\sigma}^{\mu}\sigma^{\nu}\bar{\sigma}^{\rho} = +\eta^{\mu\nu}\bar{\sigma}^{\rho}+\eta^{\nu\rho}\bar{\sigma}^{\mu}-\eta^{\mu\rho}\bar{\sigma}^{\nu}+i\varepsilon^{\mu\nu\rho\lambda}\bar{\sigma}_{\lambda}$$

$$\sigma^{\mu}\bar{\sigma}^{\nu}\sigma^{\rho} = +\eta^{\mu\nu}\sigma^{\rho}+\eta^{\nu\rho}\sigma^{\mu}-\eta^{\mu\rho}\sigma^{\nu}-i\varepsilon^{\mu\nu\rho\lambda}\sigma_{\lambda}$$

where $\varepsilon^{\mu\nu\rho\lambda}$ is the totally antisymmetric tensor with $\varepsilon^{0123}=+1$.

· Dirac spinors

A <u>Dirac spinor</u> transforms as the reducible rep. $(1/2,0)\oplus(0,1/2)$. It can be built from the dotted and undotted spinors as

$$\Psi_D = \begin{pmatrix}\psi_\alpha\\ \bar{\chi}^{\dot{\alpha}}\end{pmatrix}.$$

The Dirac matrices are given by

$$\gamma^{\mu} = \begin{pmatrix}0 & \sigma^{\mu}\\ \bar{\sigma}^{\mu} & 0\end{pmatrix} \quad , \quad \gamma_5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix}\mathbb{1} & 0\\ 0 & -\mathbb{1}\end{pmatrix}.$$

The Dirac spinor is formed by a left- and a right-handed Weyl spinor:

$$P_L\Psi_D = \frac{1+\gamma_5}{2}\Psi_D = \begin{pmatrix}\psi_\alpha\\ 0\end{pmatrix}$$

$$P_R\Psi_D = \frac{1-\gamma_5}{2}\Psi_D = \begin{pmatrix}0\\ \bar{\chi}^{\dot{\alpha}}\end{pmatrix}$$

From the 2-component spinors we can also form a <u>Majorana</u> spinor

$$\Psi_M = \begin{pmatrix}\psi_\alpha\\ \bar{\psi}^{\dot{\alpha}}\end{pmatrix}$$

by setting $\chi\equiv\psi$.

· Lagrangians for Dirac and Majorana spinors (exercises)

· We consider a Dirac fermion

$$\Psi_D = \begin{pmatrix} \psi_\alpha \\ \bar{\chi}^{\dot\alpha} \end{pmatrix} \qquad \bar{\Psi}_D = \left(\chi^\alpha \;\; \bar{\psi}_{\dot\alpha} \right)$$

whose Lagrangian is

$$\mathcal{L}_D = i \bar{\Psi}_D \gamma^\mu \partial_\mu \Psi_D - M \bar{\Psi}_D \Psi_D .$$

In 2-components notation

$$\mathcal{L}_D = i \bar{\psi} \bar{\sigma}^\mu \partial_\mu \psi + i \bar{\chi} \bar{\sigma}^\mu \partial_\mu \chi - M \left(\psi \chi + \bar{\psi} \bar{\chi} \right).$$

We also have the relations

$$\bar{\Psi}_i P_L \Psi_j = \chi_i \psi_j \qquad \bar{\Psi}_i P_R \Psi_j = \bar{\psi}_i \bar{\chi}_j$$

$$\bar{\Psi}_i \gamma^\mu P_L \Psi_j = \bar{\psi}_i \bar{\sigma}^\mu \psi_j \qquad \bar{\Psi}_i \gamma^\mu P_R \Psi_j = \chi_i \sigma^\mu \bar{\chi}_j$$

where we defined

$$\Psi_i = \begin{pmatrix} \psi_i \\ \bar{\chi}_i \end{pmatrix}$$

(i is not the 2-components index, it labels the different fermions).

· We consider a Majorana fermion

$$\Psi_M = \begin{pmatrix} \psi_\alpha \\ \bar{\psi}^{\dot\alpha} \end{pmatrix} \qquad \bar{\Psi}_M = \left(\psi^\alpha \;\; \bar{\psi}_{\dot\alpha} \right)$$

with Lagrangian

$$\mathcal{L}_M = \frac{i}{2} \bar{\Psi}_M \gamma^\mu \partial_\mu \Psi_M - \frac{1}{2} M \bar{\Psi}_M \Psi_M .$$

This can be rewritten as

$$\mathcal{L}_M = i \bar{\psi} \bar{\sigma}^\mu \partial_\mu \psi - \frac{1}{2} M \left(\psi\psi + \bar{\psi}\bar{\psi} \right).$$