XIV. INTRODUCTION TO SUPERGRAVITY

If supersymmetry is present in Nature, then for sure we will need also a supersymmetric description of gravity at least at high enough energy scales. As will be better explained before supersymmetry and gravity are deeply linked by the property that the Poincaré algebra and the supersymmetry charges are tied together in a unique superalgebra. This leads to the fact that, when we try to build a supersymmetric version of gravity, at the same time we need to construct a local version of supersymmetry, this is known as supergravity.

Supergravity plays also a fundamental role in string theory, which needs supersymmetry to allow a consistent embedding of the fermions. It can be shown that the low-energy effective description of superstring models is given by supergravity theories.

Let's try to understand better the relation between supersymmetry and gravity. First of all we recall that gravity can be seen as a theory which is invariant under local Poincaré transformations, that is, it's a gauge theory for the Poincaré group.

We can start by considering what happens if we look for a theory with local supersymmetry invariance. The anticommutation relation

$$\{Q_\alpha , \bar{Q}_{\dot\alpha}\} = 2\sigma^\mu_{\alpha\dot\alpha} P_\mu$$

implies that, on quantum fields

$$[\delta(\varepsilon_1), \delta(\varepsilon_2)] X = 2i\left(\varepsilon_2 \sigma^\mu \bar{\varepsilon}_1 - \varepsilon_1 \sigma^\mu \bar{\varepsilon}_2\right) \partial_\mu X .$$

If we try to naively extend this transformation rule to a local supersymmetry transformation by considering spacetime dependent parameters $\varepsilon_1(x)$, $\varepsilon_2(x)$ we expect that

$$[\delta(\varepsilon_1(x)), \delta(\varepsilon_2(x))] X \sim 4i\, \varepsilon_2(x) \sigma^\mu \bar{\varepsilon}_1(x)\, \partial_\mu X .$$

The right hand side of the above equation can be seen as a translation with a parameter which depends on the spacetime coordinates, this is the notion of a general coordinate transformation and leads one to expect that gravity must be present. Thus local supersymmetry should lead to gravity.

The reverse is also expected. This can be seen, for example, from the commutation rule

$$[M_{\mu\nu}, Q_\alpha] = i(\sigma_{\mu\nu})_\alpha{}^\beta Q_\beta .$$

This implies that, if we start with a supersymmetry transformation with a constant parameter ε and perform a local Lorentz transformation, then this parameter will in general become space-time dependent as a result of the Lorentz transformation. Hence local supersymmetry and gravity imply each other.

The vierbein Formalism in General Relativity

The formulation of gravity in terms of a metric tensor $g_{\mu\nu}$ is adequate for theories with fields restricted to scalars, vectors and tensors, but not for supergravity, where spinors are an indispensable ingredient. Unlike vectors and tensors, spinors have a Lorentz transformation rule that has no natural generalization to arbitrary coordinate systems. Instead, to deal with spinors, we have to introduce systems of coordinates $\xi^a_X(x)$ with $a=0,1,2,3$ that are <u>locally inertial</u> at a given point X in an arbitrary coordinate system (we can see this inertial system as the tangent space to the spacetime manifold at the point X). The principle of equivalence tells us that gravitation has no effect in these locally inertial coordinates, so the action may be expressed in terms of matter fields like spinors, vectors, etc., that are defined in these locally inertial frames. Moreover we need a <u>vierbein</u> $e^a_\mu(X)$, which arises from the transformation between the locally inertial and general coordinates

$$\underline{e^a_\mu(X) = \left.\frac{\partial \xi^a_X(x)}{\partial x^\mu}\right|_{x=X}}.$$

The action will be invariant under

- <u>general coordinate</u> transformations $x^\mu \to x'^\mu$
- <u>local Lorentz</u> transformations $\xi^a \to \xi'^a = \Lambda^a{}_b(x)\,\xi^b$ with $\Lambda^a{}_c(x)\,\Lambda^b{}_d(x)\,\eta_{ab} = \eta_{cd}$.

Under general coordinate transformations the vierbein transforms as

$$e^a{}_\mu(x) \to e'^a{}_\mu(x') = \frac{\partial x^\nu}{\partial x'^\mu}\, e^a{}_\nu(x),$$

while for a local Lorentz transformation $\xi^a(x) \to \Lambda^a{}_b(x)\,\xi^b(x)$, it transforms as

$$e^a_\mu(x) \to \Lambda^a{}_b(x)\, e^b{}_\mu(x).$$

- Theories with pure gravitation can be expressed in terms of a field, which is invariant under local Lorentz transformations and transforms as a tensor under general coordinate transformations, the metric

$$\underline{g_{\mu\nu} = e^a{}_\mu\, e^b{}_\nu\, \eta_{ab}.} \qquad (*)$$

<u>NOTE.</u> The spacetime indices $\mu,\nu,\dots$ are raised and lowered with the metric $g_{\mu\nu}$, while the Lorentz indices $a,b,\dots$ are raised and lowered with the Lorentz metric η_{ab}.

(The set of vierbeins $\{e^a_\mu(x)\}$, $a=0,1,2,3$ can be equivalently seen as a basis of tangent vectors to the point x. μ labels the components of a vector tangent to the spacetime manifold, and a is the 'name' of the vector. Condition $(*)$ implies that the vectors are orthonormal:

$$e^a_\mu(x)\, e^b_\nu(x)\, g^{\mu\nu}(x) = \eta^{ab}.$$

Local Lorentz transformations, $e^a_\mu(x) \to \Lambda^a_b(x)\, e^b_\mu(x)$, simply correspond to a change of basis in the tangent space at the point x.)

Now we have to ensure that, using the above formalism we can build theories with the wanted local invariance. Requiring local 'Poincaré' invariance is analogous to construct a gauge symmetry with the Lorentz group. This suggests that, to achieve local Lorentz invariance we need to introduce a gauge field $\omega_\mu{}^a{}_b(x)$ of the Lorentz group $SO(3,1)$. Here μ is a vector index tangent to the spacetime manifold, while a and b are $SO(3,1)$ indices. The gauge field $\omega_\mu{}^a{}_b$, which is usually called the spin connection, transforms under local Lorentz transformations in the standard way

$$\omega_\mu \to \Lambda\, \omega_\mu \Lambda^{-1} - (\partial_\mu \Lambda)\Lambda^{-1}.$$

Let's now discuss the individual choice for ω_μ which gives general relativity. Preliminarily we consider the covariant derivative of a vector field V^κ which is usually defined as

$$D_\lambda V^\kappa = \partial_\lambda V^\kappa + \Gamma^\kappa_{\lambda\nu} V^\nu ,$$

with $\Gamma^\kappa_{\lambda\nu}$ being the Christoffel symbols. On the other hand, once a vierbein is introduced one could work with $V^a(x) \equiv e^a_\mu(x) V^\mu(x)$. The V^a contains the same information as V^κ since $V^\kappa(x) = e^\kappa_a V^a(x)$. In terms of V^a the natural covariant derivative is

$$D_\mu V^a = \partial_\mu V^a + \omega_\mu{}^a{}_b V^b .$$

If we want to get the standard content of general relativity we need to ensure that

$$D_\mu V^a = e^a_\nu D_\mu V^\nu ,$$

which is guaranteed if

$$D_\mu e^a_\nu \equiv \partial_\mu e^a_\nu - \Gamma^\lambda_{\mu\nu} e^a_\lambda + \omega_\mu{}^a{}_b e^b_\nu = 0 .$$

This equation completely determines $\Gamma^\lambda_{\mu\nu}$ and $\omega_\mu{}^a{}_b$; one finds

$$\omega_\mu{}^a{}_b = \frac{1}{2} e^{\nu a}(\partial_\mu e_{b\nu} - \partial_\nu e_{b\mu}) - \frac{1}{2} e^{\nu b}(\partial_\mu e^a_\nu - \partial_\nu e^a_\mu) - \frac{1}{2} e^{\rho a} e^{\sigma b}(\partial_\rho e_{\sigma c} - \partial_\sigma e_{\rho c}) e^c_\mu ,$$

$$\Gamma^\alpha_{\mu\nu} = \frac{1}{2} g^{\alpha\lambda}(\partial_\mu g_{\nu\lambda} + \partial_\nu g_{\mu\lambda} - \partial_\lambda g_{\mu\nu}) .$$

Having defined the spin connection, we can form the gauge-covariant field strength

$$R_{\mu\nu}{}^a{}_b = \partial_\mu \omega_\nu{}^a{}_b - \partial_\nu \omega_\mu{}^a{}_b + [\omega_\mu, \omega_\nu]^a{}_b .$$

It has the same content as the Riemann tensor $R_{\mu\nu}{}^\sigma{}_\tau$ conventionally defined in terms of the Christoffel symbols and their derivative; in fact it follows that

$$R_{\mu\nu}{}^a{}_b = e^a_\sigma e^\tau_b R_{\mu\nu}{}^\sigma{}_\tau .$$

With the aid of the spin connection we can couple spinors to general relativity. As any gauge field, the spin connection can be coupled to a field $\psi(x)$ in any representation of the gauge group. Letting $\sigma^a{}_b$ be the generators of the Lorentz group in the spinor representation, the covariant derivative of ψ is

$$\underline{D_\mu \psi = \partial_\mu \psi + \frac{1}{2} \omega_\mu^{ab} \sigma_{ab} \psi .}$$

Under a local Lorentz transformation $\Lambda(x)$ we require

$$\psi(x) \to D(\Lambda(x))\, \psi(x),$$

then the covariant derivative also transforms homogeneously

$$D_\mu \psi(x) \to D(\Lambda(x))\, D_\mu \psi(x) .$$

To define the γ matrices, we first introduce the standard flat space gamma matrices γ_a that obey

$$\{\gamma_a, \gamma_b\} = 2\eta_{ab} .$$

Curved space gamma matrices are then defined as

$$\gamma_\mu(x) = e^a_\mu(x)\, \gamma_a ,$$

and they obey

$$\{\gamma_\mu(x), \gamma_\nu(x)\} = 2\, g_{\mu\nu}(x).$$

Notice that in general it is useful to consider other "non-minimal" spin connections, which normally depend on fields other than the metric. The non minimality of a spin connection is conveniently measured by the "torsion" $T^a_{\mu\nu}$ defined as

$$T^a_{\mu\nu} = D_\mu e^a_\nu - D_\nu e^a_\mu .$$

The torsion depends only on the spin connection and not on the Christoffel symbols, given that $\Gamma^\lambda_{\mu\nu} = \Gamma^\lambda_{\nu\mu}$.

Examples of theories with torsion are obtained by just adding fermions to general relativity.

· Local supersymmetry in the massless Wess–Zumino model

To understand better why local supersymmetry implies gravity it is useful to consider a simple example, the massless Wess–Zumino model.

NOTE. In this section we will adopt the 4-component spinor notation instead of the 2-compact notation used so far. We take the discussion from van Nieuwenhuizen, Phys. Repts. 68 (1981) 189–398.

The action we consider is

$$\mathcal{L} = -\frac{1}{2}(\partial_\mu A)^2 - \frac{1}{2}(\partial_\mu B)^2 - \frac{1}{2}\bar{\lambda}\not{\partial}\lambda \qquad (\text{with } \bar{\lambda} \equiv \lambda^+ \gamma^4).$$

This is invariant under the global susy transformations

$$\delta A = \frac{1}{2}\bar{\varepsilon}\lambda$$
$$\delta B = -\frac{i}{2}\bar{\varepsilon}\gamma_5\lambda \qquad (\gamma_5 \equiv \gamma^1\gamma^2\gamma^3\gamma^4).$$
$$\delta\lambda = \frac{1}{2}\not{\partial}(A - iB\gamma_5)\varepsilon$$

As we know, the Lagrangian is susy invariant apart from a total derivative:

$$\delta\mathcal{L} = \partial_\mu K^\mu = \partial_\mu\left(-\frac{1}{4}\bar{\varepsilon}\gamma^\mu[\not{\partial}(A - i\gamma_5 B)]\lambda\right).$$

We now turn to local supersymmetry and ask what happens if we make ε local, hence $\varepsilon(x)$. For example we define

$$\delta\lambda = \frac{1}{2}(\not{\partial}(A - iB\gamma_5))\,\varepsilon(x).$$

Notice that we do not want to introduce terms with $\partial_\mu\varepsilon(x)$ in the variation of the fields except for the gravitino, which correspond to a "gauge field" for the supersymmetry transformations.

We already discussed what happens for the variation of the action: it is the alternative procedure to get the supersymmetry current:

$$\delta I = \int d^4x\, (\partial_\mu\bar{\varepsilon}(x))\, j^\mu_N,$$

where j^μ_N is the supercurrent.

To cancel this variation of the action we need to introduce an extra ingredient: a gauge field which corresponds to supersymmetry. In this case the gauge parameter $\varepsilon(x)$ is a spinor (and not a scalar as in ordinary gauge theories) so we need a spinor field with an index μ as gauge field: ψ_μ. One can easily realize that such a field has spin 3/2, that is, it could be the partner of the graviton: the gravitino. We will see afterwards that this expectation is correct.

To cancel the δI variation we need a term of the form

$$I^N = \int d^4x\, (-\kappa\,\bar{\psi}_\mu\, j^\mu_N)$$

and we must require that

$$\delta\psi_\mu = \frac{1}{\varkappa}\partial_\mu\varepsilon(x) + \text{more.}$$

The coupling $\varkappa$ is a coupling with dimension of a mass.

Now, when we compute the variation of the action, we must vary also I^N. In this way we get some terms quadratic in A and B:

$$\delta(I+I^N) \supset \int d^4x\, \frac{\varkappa}{2}\left(\bar\psi_\mu\gamma_\nu\varepsilon\right)\left(T^{\mu\nu}(A) + T^{\mu\nu}(B)\right)$$

where $T_{\mu\nu}(A) = \partial_\mu A\,\partial_\nu A - \frac{1}{2}\delta_{\mu\nu}(\partial_\lambda A)^2$ is the energy-momentum tensor of the field A. The above variation can only be cancelled by adding a second coupling of a new field $g_{\mu\nu}$ to the Noether current of translations, $\frac{1}{2}T^{\mu\nu}$, and requiring that under local supersymmetry,

$$\delta g_{\mu\nu} = -\frac{\varkappa}{2}\bar\psi_\mu\gamma_\nu\varepsilon - \frac{\varkappa}{2}\bar\psi_\nu\gamma_\mu\varepsilon.$$

This tells us that we need gravity to get a theory which is locally supersymmetric invariant.

If we rewrite the theory in terms of the vierbein e^a_μ we get the following susy transformation

$$\delta e^a_\mu = \frac{1}{2}\varkappa\,\bar\varepsilon\gamma^a\psi_\mu.$$

Moreover, for the gravitino we expect

$$\delta\psi_\mu^a = \frac{1}{\varkappa}D_\mu\varepsilon, \quad \text{where} \quad D_\mu\varepsilon = \partial_\mu\varepsilon + \frac{1}{2}\omega_\mu{}^{mn}\sigma_{mn}\varepsilon.$$

To be sure that what we found is meaningful we can consider the graviton multiplet in global supersymmetry. It is given by the two physical fields e^a_μ and ψ_μ (with helicities ± 2 and $\pm 3/2$). The global susy transformation for this multiplet are

$$\delta g_{\mu\nu} = \frac{\varkappa}{2}\left(\bar\varepsilon\gamma_\mu\psi_\nu + \bar\varepsilon\gamma_\nu\psi_\mu\right), \quad \delta\psi_\mu = \frac{1}{2\varkappa}(\omega_\mu{}^{mn})_L\,\sigma_{mn}\varepsilon$$

where $(\omega_\mu{}^{mn})_L$ is a "linearized" form of the spin connection and ε is a constant spinorial parameter. This transformations exactly match with the form of the local susy transformations specialized to the case of constant ε, so we are confident that it is reasonable to identify the (ψ_μ, e^a_μ) system we found in local supersymmetry with the graviton $N=1$ supermultiplet.

- The action for simple supergravity.

As we have seen we have three objects in the pure supergravity action: the vierbein e^a_μ, the gravitino ψ_μ and the spin connection. Since e^a_μ and ψ_μ already describe the gravity supermultiplet, the field $\omega_\mu{}^{mn}$ should not be physical. Indeed $\omega_\mu{}^{mn}$ is a non-physical field, which is useful to write a simple action, but then is eliminated by solving its non-propagating equations of motion.

For pure gravity, without the gravitino, we already gave the form of the spin connection:

$$\omega_\mu{}^{mn} = \omega_\mu{}^{mn}(e).$$

As we will see, when the gravitino is introduced we get some extra ψ_μ-dependent contributions.

For the graviton we have the usual Hilbert action (with $e = \det e^m_\mu$)

$$\mathcal{L}^{(2)} = -\frac{1}{2\kappa^2}\, e\, R(e,\omega) = -\frac{1}{2\kappa^2}\, e\, e^{m\nu} e^{n\mu} R_{\mu\nu mn}(\omega).$$

If we solve for ω we get the $\omega_\mu{}^{mn}(e)$ given in the previous discussion on general relativity.

We now turn to the fermionic part of the action. This is given by the Rarita-Schwinger action in curved space

$$\mathcal{L}^{(3/2)} = -\frac{1}{2}\, \varepsilon^{\mu\nu\rho\sigma}\, \bar{\psi}_\mu \gamma_5 \gamma_\nu D_\rho \psi_\sigma,$$

where we have

$$D_\rho \psi_\sigma = \left(\partial_\rho + \frac{1}{2}\omega_\rho{}^{mn}\sigma_{mn}\right)\psi_\sigma .$$

Notice that we did not include a term with $\Gamma^\mu_{\nu\rho}$ in the expression for $D_\rho\psi_\sigma$, because in any case it would have been cancelled by the $\varepsilon^{\mu\nu\rho\sigma}$ factor.

One can check that the total action

$$\mathcal{L} = \mathcal{L}^{(2)} + \mathcal{L}^{(3/2)}$$

is invariant under the local supersymmetry transformations

$$\begin{cases} \delta e^m_\mu = \frac{\kappa}{2}\,\bar{\varepsilon}\,\gamma^m \psi_\mu \\ \delta\psi_\mu = \frac{1}{\kappa} D_\mu \varepsilon \end{cases}.$$

As we already anticipated, the spin connection obtained by solving the equations of motion contains a term which depends on the gravitino:

$$\omega_{\mu mn}(e,\psi) = \omega_{\mu mn}(e) + \frac{\kappa^2}{4}\left(\bar{\psi}_\mu \gamma_m \psi_n - \bar{\psi}_\mu \gamma_n \psi_m + \bar{\psi}_m \gamma_\mu \psi_n\right).$$

· Auxiliary fields for the supergravity action.

Auxiliary fields are needed if we want the supersymmetry algebra to close also off-shell. Moreover in supergravity they are needed in order that the transformation rules of the graviton supermultiplet do not depend on matter fields. If they did, without further modifications, two matter actions, each of which has been coupled to gravity in an invariant way, could not be put together. The reason is that the field transformation rules of system I would not work for system II and viceversa.

However, if one adds auxiliary fields, the field transformation rules are always the same, independent of the matter fields and valid for any matter coupling system.

Let's start by counting how many auxiliary fields we need. In supergravity there are three local invariances:

- general coordinate transformations G
- local Lorentz rotations L
- local supersymmetry transformations Q

Thus the counting of field components in the pure supergravity action is

$$16\, e^m{}_\mu - 4 \text{ gen. coord.} - 6 \text{ local Lorentz} = 6 \text{ bosonic fields.}$$

$$16\, \psi^a_\mu - 4 \text{ local supersymmetry} = 12 \text{ fermionic fields.}$$

Hence there is a mismatch of 6 bosonic components. The algebra is thus not closed and we need to add $6+n$ bosonic auxiliary fields and n fermionic auxiliary fields. There exist several sets of auxiliary fields, the most prominent one is the minimal set with $n=0$, consisting of an <u>axial vector A_m</u>, a <u>scalar S</u> and a <u>pseudoscalar P</u>.

The action is

$$\mathcal{L} = \mathcal{L}^{(2)}(e,\omega) + \mathcal{L}^{(3/2)}(e,\psi,\omega) - \frac{e}{3}\left(S^2 + P^2 - A_m^2\right),$$

so S, P and A_m are non-propagating fields, as expected. This action is invariant under the local supersymmetry transformations

$$\delta e^m{}_\mu = \frac{\varkappa}{2}\bar{\varepsilon}\gamma^m\psi_\mu$$

$$\delta\psi_\mu = \frac{1}{\varkappa}\left(D_\mu + i\frac{\varkappa}{2}A_\mu\gamma_5\right)\varepsilon + \frac{1}{6}\gamma_\mu\left(S - i\gamma_5 P - i\not{A}\gamma_5\right)\varepsilon$$

$$\delta S = \frac{1}{4}\bar{\varepsilon}\gamma\cdot R^{cov}$$

$$\delta P = -\frac{i}{4}\bar{\varepsilon}\gamma_5\gamma\cdot R^{cov}$$

$$\delta A_m = \frac{3i}{4}\bar{\varepsilon}\gamma_5\left(R_m^{cov} - \frac{1}{3}\gamma_m\gamma\cdot R^{cov}\right)$$

where R_μ^{cov} is the gravitino field equation ($R^\mu = \varepsilon^{\mu\nu\rho\sigma}\gamma_5\gamma_\nu D_\rho\psi_\sigma$) but with the supercovariant derivatives:

$$R^{\mu,cov} = \varepsilon^{\mu\nu\rho\sigma}\gamma_5\gamma_\nu\left(D_\rho\psi_\sigma - \frac{i}{2}A_\sigma\gamma_5\psi_\rho - \frac{1}{6}\gamma_\sigma\left(S - i\gamma_5 P - i\not{A}\gamma_5\right)\psi_\rho\right).$$

As expected, with auxiliary fields the algebra closes. In this case the $\{Q, Q\}$ anticommutator can be written in a form valid for all fields

$$[\delta_Q(\epsilon_1), \delta_Q(\epsilon_2)] = \delta_{G}(\xi^a) + \delta_Q(-\xi^a \psi_a) + \delta_L\left[\xi^\kappa \hat{\omega}_\mu{}^{mn} + \frac{1}{3}\bar{\epsilon}_2 \sigma^{mn}(S - i\gamma_5 P)\epsilon_1\right]$$

where

$$\hat{\omega}_{\mu ab} = \omega_{\mu ab} - \frac{i}{3} \epsilon_{\mu abc} A^c \,,$$

$$\xi^\kappa = \frac{1}{2} \bar{\epsilon}_2 \gamma^\kappa \epsilon_1 \,.$$

In the above expressions δ_Q denotes a supersymmetry transformation, δ_G a general coordinate transformation and δ_L a local Lorentz transformation with the parameters given in brackets.
Notice that the parameters of the transformation depend on the fields (and auxiliary fields), this is a peculiar property of supergravity, which is not present in gravity or Yang Mills theories.