XI. N=2 GLOBALLY SUPERSYMMETRIC ACTIONS

In this section we will discuss the simplest examples of theories with extended global supersymmetry, namely the $N=2$ globally supersymmetric actions. Theories with unbroken extended supersymmetry are not useful to build realistic extensions of the Standard Model, because they can not give a chiral spectrum. Nevertheless gauge theories with extended susy are interesting because they have provided examples for the use of powerful mathematical methods to solve dynamical problems. Moreover extended susy theories present very peculiar features under renormalization and even give the possibility to build completely finite quantum field theory models.

There are some special formalisms that have been proposed to construct Lagrangians with $N=2$ supersymmetry, but we can also build them using the $N=1$ formalism we already know. Any theory with $N=2$ supersymmetry has also an $N=1$ supersymmetry, so its Lagrangian must be a special case of the Lagrangians we already constructed for $N=1$ susy. To build an $N=2$ susy Lagrangian we need only to write down the most general Lagrangian with $N=1$ susy whose $N=1$ supermultiplets contain physical fields for the particles in the $N=2$ supermultiplets, and then impose a discrete R-symmetry on the Lagrangian. The Lagrangian will then be invariant under a second supersymmetry, whose supermultiplets are given by acting on the original $N=1$ supermultiplets with the R-symmetry.

It is sufficient to chose the discrete R-transformation so that

$$Q^1_\alpha \to Q^2_\alpha \quad , \quad Q^2_\alpha \to -Q^1_\alpha .$$

If the central charges were zero the susy algebra would be invariant under an $SU(2)$ R-symmetry group (our discrete transformation is just the element $\exp(i\pi\sigma_2/2)$), but symmetry under the discrete symmetry is enough to build an $N=2$ theory, so we do not need to assume that the central charges are zero. However we will see that the Lagrangian we will build by this method will have an $SU(2)$ R-symmetry, and not just the discrete symmetry we considered.

• The vector multiplet Lagrangian

We saw that in $N=2$ a massless gauge boson belongs to a multiplet also containing a pair of massless fermions that transform as a doublet under the $SU(2)$ R-symmetry and a pair of real $SU(2)$-singlet scalars. In an $N=1$ language, this multiplet is clearly formed by a vector multiplet V and a chiral multiplet Φ. However, now all the fields are in the same $N=2$ multiplet, so they must be in the same representation of the gauge group, namely the adjoint representation. The two supermultiplets have the components

$$V^a = (v_\mu^a,\ \lambda^a,\ D^a)$$

and

$$\Phi^a = (z^a,\ \psi^a,\ F^a)\ .$$

Under the discrete R-transformation

$$\psi^a \to \lambda^a \quad \text{and} \quad \lambda^a \to -\psi^a\ .$$

The $N=1$ Lagrangian for the chiral multiplet is (we explicitly write the coupling constant g)

$$\mathcal{L}^{N=1}_{matter} = \int d^2\theta\, d^2\bar\theta\ \mathrm{Tr}\ \bar\Phi\, e^{2gV}\,\Phi = \mathrm{Tr}\Big[(D_\mu z)^+ D^\mu z - i\psi\sigma^\mu D_\mu\bar\psi + F^+F$$
$$+ i\sqrt{2}g\, z^+\{\lambda,\psi\} - i\sqrt{2}g\,\{\bar\psi,\bar\lambda\}z + g D[z,z^+]\Big]$$

where we defined

$$z = z^a T^a\ ,\quad \psi = \psi^a T^a\ ,\quad F = F^a T^a\ ,\quad a = 1,\dots,\dim G$$

in addition to

$$\lambda = \lambda^a T^a\ ,\quad D = D^a T^a\ ,\quad v_\mu = v_\mu^a T^a$$

and

$$\begin{cases} D_\mu z = \partial_\mu z - i g v_\mu^a T^a z \\ D_\mu \psi = \partial_\mu\psi - i g v_\mu^a T^a \psi \end{cases}$$

The commutators and anticommutators arise since the generators in the adjoint representation are given by

$$(T^a_{adj})_{bc} = -i f_{abc}$$

and we normalize the generators as

$$\mathrm{Tr}\ T^a T^b = \delta^{ab}$$

so that

$$z^+\lambda\psi \to z_b^+ \lambda^a (T^a_{adj})_{bc}\psi^c = -i z_b^+\lambda^a f_{abc}\psi^c = i z_b^+ f_{bac}\lambda^a\psi^c$$
$$= z_b^+\lambda^a\psi^c\,\mathrm{Tr}\ T^b[T^a,T^c] = \mathrm{Tr}\ z^+\{\lambda,\psi\}$$

and

$$z^+ D z \to z_b^+ D^a (T^a_{adj})_{bc} z^c = -i f_{abc} z_b^+ D^a z^c = -\mathrm{Tr}\ D[z^+,z] = \mathrm{Tr}\ D[z,z^+]\ .$$

We must now add the action for the gauge superfield $\mathcal{L}^{N=1}_{gauge}$.

We get

$$\mathcal{L}_{YM}^{N=2} = \frac{1}{32\pi}\,\mathrm{Im}\left(\tau \int d^2\theta\, \mathrm{Tr}\, W^a W_a\right) + \int d^2\theta\, d^2\bar\theta\, \mathrm{Tr}\, \bar\Phi\, e^{2gV}\, \Phi$$

$$= \mathrm{Tr}\Big(-\frac{1}{4} F_{\mu\nu}F^{\mu\nu} - i\lambda\sigma^\mu D_\mu \bar\lambda - i\psi\sigma^\mu D_\mu\bar\psi + (D_\mu z)^+ D^\mu z$$

$$+ \frac{\theta}{32\pi^2} g^2\, \mathrm{Tr}\, F_{\mu\nu}\tilde F^{\mu\nu} + \frac{1}{2}D^2 + F^+F$$

$$+ i\sqrt{2}\, g\, z^+\{\lambda,\psi\} - i\sqrt{2}\, g\,\{\bar\psi,\bar\lambda\} z + g\, D[z, z^+] \Big).$$

As one can simply check, the relative coefficients of $\mathcal{L}^{N=1}_{gauge}$ and $\mathcal{L}^{N=1}_{matter}$ in the above Lagrangian have been chosen in such a way to respect the discrete R-transformation: the λ and ψ kinetic terms have the same coefficient and the Yukawa couplings $z^+\{\lambda,\psi\}$ and $\{\bar\psi,\bar\lambda\}z$ also exhibit the symmetry. Notice that the Lagrangian has an even larger invariance: it is invariant under the $SU(2)$ R-symmetry under which the fermions ψ and λ form a doublet (while the other fields are singlets).

The Lagrangian $\mathcal{L}_{YM}^{N=2}$ has an $N=1$ supersymmetry with multiplets

$$(z^a, \psi^a, F^a) \quad \text{and} \quad (v_\mu^a, \lambda^a, D^a)$$

and also a <u>second</u> independent $N=1$ supersymmetry with multiplets

$$(z^a, \lambda^a, F^a) \quad \text{and} \quad (v_\mu^a, -\psi^a, D^a),$$

it therefore satisfies the conditions imposed by $N=2$ supersymmetry.

Notice that we <u>can not add a superpotential</u> term for Φ, because it would give ψ^a interactions or mass terms that are absent for λ^a.

The auxiliary fields equations are

$$F^a = 0$$

$$D^a = -g\,[z, z^+]^a$$

leading to a scalar potential

$$V(z, z^+) = \frac{1}{2} g^2\, \mathrm{Tr}\left([z, z^+]\right)^2.$$

This potential has a minimum value of zero, which is reached, not only for $z^a = 0$, but also for any set of z for which

$$\Big[\sum_a t^a\, \mathrm{Re}\, z^a, \sum_b t^b\, \mathrm{Im}\, z^b\Big] = 0.$$

That is, the minimum of the potential is reached for those scalar fields for which all generators $\sum_a t^a\, \mathrm{Re}\, z^a$ and $\sum_a t^a\, \mathrm{Im}\, z^a$ belong to a Cartan subalgebra of the full gauge algebra, all of whose generators commute with one another.

All such values of z give zero potential, and hence unbroken $N=2$ supersymmetry, but they are not physically equivalent, as shown for instance by the different masses they give to the gauge bosons associated with the broken gauge symmetries.

An important property of the massive states associated to the breaking of the gauge invariance is the fact that they are necessarily given by short $N=2$ supermultiplets, that is their masses saturate the bound given by the $N=2$ central charge. This can be easily checked by counting the degrees of freedom associated to these states.
First of all we recall that, even if the gauge symmetry is broken, susy is unbroken, so all the states must fill complete $N=2$ supermultiplets. Given that the breaking of gauge invariance is spontaneous, the number of degrees of freedom associated to the fields is exactly the same as for the gauge-invariant vacuum, or, in other words, the number of degrees of freedom is the same as for massless $N=2$ vector supermultiplets. The states which get a mass from gauge breaking will be contained in massive $N=2$ vector supermultiplets, but the only massive multiplets which have the same number of physical states as the massless multiplets are the short supermultiplets. (Recall that long massive supermultiplets $N=2$ have 4 times as many states as the corresponding massless multiplets.)

One can also explicitly verify that the massive states form short multiplets by computing the central charges for these multiplets. There is a nice way to use the supercurrents to extract the central charges (see Weinberg III section 27.9. for a detailed discussion), the final result for the SU(2) case is

$$Z = 2\sqrt{2}\, v\, [q + i g]$$

where $v = \langle z_3 \rangle$ is the VEV which breaks the SU(2) symmetry to U(1) and q and g are the electric and magnetic charges with respect to the unbroken U(1):

$$q = \int dS_i E^i \qquad , \qquad g = \int dS_i B^i$$

(the integral is on an S_2 surface at spatial infinity).

By computing the masses of the states, one gets a massless multiplet related to the unbroken U(1) and a pair of massive multiplets with U(1) charges $\pm e$ (corresponding to the broken gauge generators) whose masses are given by

$$m = \sqrt{2}\, |e v|$$

and whose magnetic charge is zero. For these states we saturate the BPS bound

$$m = \frac{1}{2} |Z| ,$$

so they live in short $N=2$ supermultiplets.

NOTE. If one relaxes the renormalizability requirement, a much more general action for $N=2$ vector supermultiplets can be written. It turns out that its general form is

$$\mathcal{L} = \frac{1}{16\pi i}\int d^2\theta\ \mathcal{F}_{ab}(\Phi)\, W^{a\alpha} W^b{}_\alpha + \frac{1}{32\pi i}\int d^2\theta\, d^2\bar\theta\ (\bar\Phi e^{2V})^a \mathcal{F}_a(\Phi) + h.c.$$

where $\mathcal{F}_{ab}$ and $\mathcal{F}_a$ are given by

$$\mathcal{F}_a(\Phi) = \frac{\partial \mathcal{F}(\Phi)}{\partial \Phi_a},$$

$$\mathcal{F}_{ab}(\Phi) = \frac{\partial^2 \mathcal{F}(\Phi)}{\partial\Phi_a\, \partial\Phi_b}.$$

$\mathcal{F}(\Phi)$ is an holomorphic functional called the $N=2$ prepotential and it determines the complete Lagrangian. (choosing $\mathcal{F} = \frac{\tau}{2}\,\mathrm{tr}\,\Phi^2$ we get the original renormalizable $N=2$ Lagrangian.)

· The hypermultiplet Lagrangian

In $N=2$ theories we can also introduce hypermultiplets in the Lagrangian. In terms of $N=1$ superfields, an hypermultiplet is built by taking two chiral superfields

$$H_1 = (H^+, \psi_\alpha^+, F^+),$$

$$H_2 = (H^-, \psi_\alpha^-, F^-).$$

The scalar components H^+, H^- form an $SU(2)$ doublet, while the spinors $\psi_\alpha^\pm$ are singlets. We can build an $N=2$ Lagrangian by imposing the discrete R-transformation

$$\begin{cases} H^+ \to -(H^-)^* \\ H^- \to (H^+)^* \end{cases}$$

together with the transformation for the spinor components of the vector supermultiplet ($\psi^a \to \lambda^a$, $\lambda^a \to -\psi^a$). The Lagrangian is given by

$$\mathcal{L}^{N=2}_{hyper} = \int d^2\theta\, d^2\bar\theta\ (\bar H_1 e^{2V} H_1 + \bar H_2 e^{2V} H_2) + \int d^2\theta\ \bar H_1 \Phi H_2 + h.c.$$

One also finds that $N=2$ supersymmetry requires H_1 and H_2 to be in complex conjugate representations of the gauge group.

Note: A mass term $\mu H^+ H^-$ can be introduced in the $N=2$ Lagrangian for the hypermultiplet.

· $N=4$ supersymmetric theories

We can construct an $N=4$ susy theory by considering an $N=2$ Lagrangian which contains a vector multiplet an an hypermultiplet both in the adjoint representation of the gauge group. Moreover no mass term is allowed for the hypermultiplet. To get an $N=4$ susy we also must impose an $SU(4)$ R-symmetry. Notice that $N=4$ theories have essentially only one free parameter given by the gauge coupling.

(see Weinberg III section 27.9. for a detailed discussion.)