Quantum Field Theory III

HS 10, Exercise Sheet 1

Due date: 29.09.2010

Exercise 1:
Show that for two 2-components spinors ψ and χ, we have

a) $\psi\chi = \chi\psi$

b) $\bar{\psi}\bar{\chi} = \bar{\chi}\bar{\psi}$

c) $(\psi\chi)^\dagger = \bar{\psi}\bar{\chi}$

d) $\bar{\psi}\bar{\sigma}^\mu\chi = -\chi\sigma^\mu\bar{\psi} = (\bar{\chi}\bar{\sigma}^\mu\psi)^\star = -(\psi\sigma^\mu\bar{\chi})^\star$

e) $\psi\sigma^\mu\bar{\sigma}^\nu\chi = \chi\sigma^\nu\bar{\sigma}^\mu\psi = (\bar{\chi}\bar{\sigma}^\nu\sigma^\mu\bar{\psi})^\star = (\bar{\psi}\bar{\sigma}^\mu\sigma^\nu\bar{\chi})^\star$

Exercise 2:
Prove the Fierz rearrangement identity:

$$\chi_\alpha(\xi\eta) = -\xi_\alpha(\eta\chi) - \eta_\alpha(\chi\xi).$$

Exercise 3:
Prove the following reduction identities:

a) $\sigma^\mu_{\alpha\dot{\alpha}}\bar{\sigma}^{\dot{\beta}\beta}_\mu = 2\delta^\beta_\alpha\delta^{\dot{\beta}}_{\dot{\alpha}}$

b) $\sigma^\mu_{\alpha\dot{\alpha}}\sigma_{\mu,\beta\dot{\beta}} = 2\epsilon_{\alpha\beta}\epsilon_{\dot{\alpha}\dot{\beta}}$

c) $\bar{\sigma}^{\mu\dot{\alpha}\alpha}\bar{\sigma}^{\dot{\beta}\beta}_\mu = 2\epsilon^{\alpha\beta}\epsilon^{\dot{\alpha}\dot{\beta}}$

d) $[\sigma^\mu\bar{\sigma}^\nu + \sigma^\nu\bar{\sigma}^\mu]_\alpha{}^\beta = 2\eta^{\mu\nu}\delta^\beta_\alpha$

e) $[\bar{\sigma}^\mu\sigma^\nu + \bar{\sigma}^\nu\sigma^\mu]^{\dot{\beta}}{}_{\dot{\alpha}} = 2\eta^{\mu\nu}\delta^{\dot{\beta}}_{\dot{\alpha}}$

f) $\bar{\sigma}^\mu\sigma^\nu\bar{\sigma}^\rho = \eta^{\mu\nu}\bar{\sigma}^\rho + \eta^{\nu\rho}\bar{\sigma}^\mu - \eta^{\mu\rho}\bar{\sigma}^\nu + i\epsilon^{\mu\nu\rho\lambda}\bar{\sigma}_\lambda$

g) $\sigma^\mu\bar{\sigma}^\nu\sigma^\rho = \eta^{\mu\nu}\sigma^\rho + \eta^{\nu\rho}\sigma^\mu - \eta^{\mu\rho}\sigma^\nu - i\epsilon^{\mu\nu\rho\lambda}\sigma_\lambda$,

where $\epsilon^{\mu\nu\rho\lambda}$ is the totally antisymmetric tensor with $\epsilon^{0123} = +1$.

Exercise 4:
In this exercise we will learn how 2-components spinors are connected to the Dirac spinors we used in QFT 1.
A **Dirac spinor** transforms in the reducibel representation $(1/2, 0) \oplus (0, 1/2)$. It can be built from the dotted and undotted spinors as

$$\Psi_D = \begin{pmatrix} \psi_\alpha \\ \bar{\chi}^{\dot{\alpha}} \end{pmatrix}.$$

The Dirac gamma matrices are given by

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}, \qquad \gamma_5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 1_{2x2} & 0 \\ 0 & -1_{2x2} \end{pmatrix}.$$

The Dirac spinor is formed by a left- and a right-handed Weyl spinor:

$$P_L\Psi_D = \frac{1+\gamma_5}{2}\Psi_D = \begin{pmatrix} \psi_\alpha \\ 0 \end{pmatrix}$$
$$P_R\Psi_D = \frac{1-\gamma_5}{2}\Psi_D = \begin{pmatrix} 0 \\ \bar{\chi}^{\dot{\alpha}} \end{pmatrix}$$

From the 2-components spinors we can also form a **Majorana** spinor

$$\Psi_M = \begin{pmatrix} \psi_\alpha \\ \bar{\psi}^{\dot{\alpha}} \end{pmatrix}$$

by setting $\chi \equiv \psi$.

a) Show that the Lagrangian for a Dirac fermion

$$\mathcal{L}_D = i\bar{\Psi}_D\gamma^\mu\partial_\mu\Psi_D - M\bar{\Psi}_D\Psi_D$$

written in 2-components notation up to a total divergence is

$$\mathcal{L}_D = i\bar{\psi}\bar{\sigma}^\mu\partial_\mu\psi + i\bar{\chi}\bar{\sigma}^\mu\partial_\mu\chi - M(\psi\chi + \bar{\psi}\bar{\chi}).$$

Note that $\bar{\Psi}_D = (\chi^\alpha \quad \bar{\psi}_{\dot{\alpha}})$.

b) Prove the following identities:
$\overline{\Psi}_i P_L \Psi_j = \chi_i\psi_j$

c) $\overline{\Psi}_i P_R \Psi_j = \bar{\psi}_i\bar{\chi}_j$

d) $\overline{\Psi}_i \gamma^\mu P_L \Psi_j = \bar{\psi}_i\bar{\sigma}^\mu\psi_j$

e) $\overline{\Psi}_i \gamma^\mu P_R \Psi_j = \chi_i\sigma^\mu\bar{\chi}_j$

f) Show that the Lagrangian for Majorana fermions

$$\mathcal{L}_M = \frac{i}{2}\bar{\Psi}_M\gamma^\mu\partial_\mu\Psi_M - \frac{1}{2}M\bar{\Psi}_M\Psi_M$$

written in 2-components notation up to a total divergence is

$$\mathcal{L}_M = i\bar{\psi}\bar{\sigma}^\mu\partial_\mu\psi - \frac{1}{2}M(\psi\psi + \bar{\psi}\bar{\psi}).$$