Chapter 7

Unitary symmetries and QCD as a gauge theory

Literature:

- Lipkin [23] (group theory concepts from a physicist's point of view)
- Lee [24], chapter 20 (extensive treatment of Lie groups and Lie algebras in the context of differential geometry)

Interactions between particles should respect some observed symmetry. Often, the procedure of postulating a specific symmetry leads to a unique theory. This way of approach is the one of **gauge theories**. The usual example of a gauge theory is QED, which corresponds to a local $U(1)$-symmetry of the Lagrangian :

$$\psi \to \psi' = \mathrm{e}^{ieq_e\chi(x)}\psi, \tag{7.1}$$

$$A_\mu \to A'_\mu = A_\mu - \partial_\mu \chi(x). \tag{7.2}$$

We can code this complicated transformation behavior by replacing in the QED Lagrangian ∂_μ by the **covariant derivative** $D_\mu = \partial_\mu + ieq_e A_\mu$.

7.1 Isospin $SU(2)$

For this section we consider only the strong interaction and ignore the electromagnetic and weak interactions. In this regard, isobaric nuclei (with the same mass number A) are very similar. Heisenberg proposed to interpret protons and neutrons as two states of the same object : the **nucleon**:

$$|p\rangle = \psi(x)\begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

$$|n\rangle = \psi(x)\begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

We note the analogy to the spin formalism of nonrelativistic quantum mechanics, which originated the name *isospin*.

In isospin-space, $|p\rangle$ and $|n\rangle$ can be represented as a two-component spinor with $I = \frac{1}{2}$. $|p\rangle$ has then $I_3 = +\frac{1}{2}$ and $|n\rangle$ has $I_3 = -\frac{1}{2}$.

Since the strong interaction is blind to other charges (electromagnetic charge, weak hypercharge), the (strong) physics must be the same for any linear combinations of $|p\rangle$ and $|n\rangle$. In other words, for,

$$\begin{aligned} |p\rangle \to |p'\rangle &= \alpha\,|p\rangle + \beta\,|n\rangle\,, \\ |n\rangle \to |n'\rangle &= \gamma\,|p\rangle + \delta\,|n\rangle\,, \end{aligned}$$

for some $\alpha, \beta, \gamma, \delta \in \mathbb{C}$, or,

$$|N\rangle = \begin{pmatrix} \psi_p \\ \psi_n \end{pmatrix} \to |N'\rangle = U\,|N\rangle\,, \tag{7.3}$$

for some 2×2 matrix U with complex entries, the (strong) physics does not change if we switch from $|N\rangle$ to $|N'\rangle$ to describe the system.

We remark at this point that this symmetry is only an *approximate* symmetry since it is violated by the other interactions, and is hence *not* a symmetry of nature.

First we require the conservation of the norm $\langle N|N\rangle$ which we interpret as the number of particles like in quantum mechanics. This yields,

$$\begin{aligned} \langle N|N\rangle \to \langle N'|N'\rangle = \langle N|\,U^\dagger U|N\rangle &\overset{!}{=} \langle N|N\rangle \\ \Rightarrow U^\dagger U = UU^\dagger &= \mathbb{1} \Rightarrow U \in U(2). \end{aligned} \tag{7.4}$$

A general unitary matrix has 4 real parameters. Since the effect of U and $\mathrm{e}^{i\varphi}U$ are the same, we fix one more parameter by imposing,

$$\det U \overset{!}{=} 1 \Rightarrow U \in SU(2), \tag{7.5}$$

the special unitary group in 2 dimensions. This group is a **Lie group** (a group which is at the same time a manifold). We use the representation,

$$U = \mathrm{e}^{i\alpha_j \hat{I}_j}, \tag{7.6}$$

where the α_j's are arbitrary group parameters (constant, or depending on the spacetime coordinate x), and the $\hat{I}_j$'s are the generators of the Lie group.

We concentrate on infinitesimal transformations, for which $\alpha_j \ll 1$. In this approximation we can write

$$U \approx \mathbb{1} + i\alpha_j \hat{I}_j. \tag{7.7}$$

The two defining conditions of $SU(2)$, Eq. (7.4) and (7.5), imply then for the generators,

$$\hat{I}_j^\dagger = \hat{I}_j \qquad (hermitian), \tag{7.8}$$
$$\mathrm{Tr}\,\hat{I}_j = 0 \qquad (traceless). \tag{7.9}$$

In order for the exponentiation procedure to converge for noninfinitesimal α_j's, the generators must satisfy a comutation relation, thus defining the **Lie algebra** $su(2)$ of the group $SU(2)$.

Quite in general, the commutator of two generators must be expressible as a linear combination of the other generators [1]. In the case of $su(2)$ we have,

$$[\hat{I}_i, \hat{I}_j] = i\varepsilon_{ijk}\hat{I}_k, \tag{7.10}$$

where ε_{ijk} is the totally antisymmetric tensor with $\varepsilon_{123} = +1$. They are characteristic of the (universal covering group of the) Lie group (but independent of the chosen representation) and called structure constants of the Lie group.

The representations can be characterized according to their total isospin. Consider now $I = 1/2$, where the generators are given by

$$\hat{I}_i = \frac{1}{2}\tau_i$$

with $\tau_i = \sigma_i$ the Pauli spin matrices (this notation is chosen to prevent confusion with ordinary spin):

$$\tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \tau_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

which fulfill $[\sigma_i, \sigma_j] = 2i\varepsilon_{ijk}\sigma_k$. The action of the matrices of the representation (see Eq. (7.6)) is a non-abelian phase transformation:

$$|N'\rangle = e^{i\vec{\alpha}\cdot\frac{\vec{\tau}}{2}}\,|N\rangle\,.$$

For $SU(2)$, there exists only one diagonal matrix (τ_3). In general, for $SU(N)$, the following holds true:

- *Rank* $r = N - 1$: There are r simultaneously diagonal operators.
- *Dimension* of the Lie algebra $o = N^2 - 1$: There are o generators of the group and therefore o group parameters. E. g. in the case of $SU(2)/\{\pm\mathbb{1}\} \cong SO(3)$ this means that there are three rotations/generators and three angles as parameters.

[1] Since we are working in a matrix representation of $SU(2)$ this statement makes sense. The difference between the abstract group and its matrix representation is often neglected.

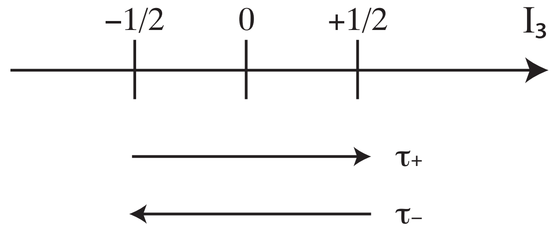


Figure 7.1: *The nucleons $|n\rangle$ and $|p\rangle$ form an isospin doublet.*

Isospin particle multiplets (representations) can be characterized by their quantum numbers I and I_3: There are $2I+1$ states. Consider for example once again the case $I = 1/2$. There are two states, characterized by their I_3 quantum number:

$$\begin{pmatrix} \left|I = \frac{1}{2}, I_3 = +\frac{1}{2}\right\rangle \\ \left|I = \frac{1}{2}, I_3 = -\frac{1}{2}\right\rangle \end{pmatrix} = \begin{pmatrix} |p\rangle \\ |n\rangle \end{pmatrix}.$$

This is visualized in Fig. 7.1, along with the action of the operators $\tau_\pm = 1/2(\tau_1 \pm i\tau_2)$:

$$\begin{aligned} &\tau_- |p\rangle = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = |n\rangle \\ &\tau_+ |n\rangle = |p\rangle \\ &\tau_- |n\rangle = \tau_+ |p\rangle = 0. \end{aligned}$$

This is the smallest non-trivial representation of $SU(2)$ and therefore its fundamental representation.

Further examples for isospin multiplets are

I	multiplets	I_3
$\frac{1}{2}$	$\begin{pmatrix} p \\ n \end{pmatrix}$ $\begin{pmatrix} K^+ \\ K^0 \end{pmatrix}$ $\begin{pmatrix} {}^3_2\mathrm{He} \\ {}^3_1\mathrm{H} \end{pmatrix}$	$+\frac{1}{2}$ $-\frac{1}{2}$
1	$\begin{pmatrix} \pi^+ \\ \pi^0 \\ \pi^- \end{pmatrix}$	$+1$ 0 -1
$\frac{3}{2}$	$\begin{pmatrix} \Delta^{++} \\ \Delta^+ \\ \Delta^0 \\ \Delta^- \end{pmatrix}$	$+\frac{3}{2}$ $+\frac{1}{2}$ $-\frac{1}{2}$ $-\frac{3}{2}$

where $m_\Delta \approx 1232\,\mathrm{MeV}$ and $m_{p,n} \approx 938\,\mathrm{MeV}$.

All $I \geqslant 1$ representations can be obtained from direct products out of the fundamental $I = 1/2$ representation $\underline{2}$ where "2" denotes the number of states. In analogy to the addition of two electron spins where the Clebsch-Gordan decomposition reads rep.$_{1/2}$ $\otimes$

$\text{rep.}_{1/2} = \text{rep.}_0 \oplus \text{rep.}_1$ and where there are two states for the spin-1/2 representation, one state for the spin-0 representation, and three states for the spin-1 representation, we have

$$\underbrace{\underline{2} \otimes \underline{2}}_{I=|\frac{1}{2}\pm\frac{1}{2}|=0,1} \quad = \quad \underbrace{\underline{1}}_{\text{isosinglet, } I=0} \quad \oplus \quad \underbrace{\underline{3}}_{\text{isotriplet, } I=1} \quad . \tag{7.11}$$

However, there is an important difference between isospin and spin multiplets. In the latter case, we are considering a bound system and the constituents carrying the spin have the same mass. On the other hand, pions are not simple bound states. Their structure will be described by the quark model.

7.1.1 Isospin invariant interactions

Isospin invariant interactions can be constructed by choosing $SU(2)$ invariant interaction terms $\mathcal{L}'$. For instance, consider the Yukawa model, describing nucleon-pion coupling, where

$$\mathcal{L}'_{\pi N} = ig\bar{N}\vec{\tau}N \cdot \vec{\pi} = ig\bar{N}'\vec{\tau}N' \cdot \vec{\pi}' \tag{7.12}$$

which is an isovector and where the second identity is due to $SU(2)$ invariance. Infinitesimally, the transformation looks as follows:

$$N' = UN \qquad U = \mathbb{1} + \frac{i}{2}\vec{\alpha} \cdot \vec{\tau} \tag{7.13}$$

$$\bar{N}' = \bar{N}U^\dagger \qquad U^\dagger = \mathbb{1} - \frac{i}{2}\vec{\alpha} \cdot \vec{\tau} = U^{-1} \tag{7.14}$$

$$\vec{\pi}' = V\vec{\pi} \qquad V = \mathbb{1} + i\vec{\alpha} \cdot \vec{t}. \tag{7.15}$$

The parameters $\vec{t}$ can be determined from the isospin invariance condition in Eq. (7.12):

$$\bar{N}\tau_j N\pi_j = \bar{N}U^{-1}\tau_i UNV_{ij}\pi_j.$$

With $V_{ij} = \delta_{ij} + i\alpha_k(t_k)_{ij}$ (cp. Eq. (7.15)) and inserting the expressions for U and $U^\dagger$, this yields

$$\begin{aligned}
\tau_j &= \underbrace{\left(\mathbb{1} - \frac{i}{2}\alpha_k\tau_k\right)\tau_i\left(\mathbb{1} + \frac{i}{2}\alpha_k\tau_k\right)}_{\substack{= \tau_i + \frac{i}{2}\alpha_k[\tau_i,\tau_k] + \mathcal{O}(\alpha_k^2) \\ = \tau_i + \frac{i}{2}\alpha_k 2i\varepsilon_{ikl}\tau_l + \mathcal{O}(\alpha_k^2)}}\left(\delta_{ij} + i\alpha_k(t_k)_{ij}\right) \\
&= \tau_j + i\alpha_k\left\{i\varepsilon_{jkl}\tau_l + \tau_i(t_k)_{ij}\right\} \\
&= \tau_j + i\alpha_k\tau_i\underbrace{\left\{i\varepsilon_{jki} + (t_k)_{ij}\right\}}_{\overset{!}{=}0} \\
&\Rightarrow (t_k)_{ij} = -i\varepsilon_{kij}.
\end{aligned}$$

This means that the 3×3 matrices t_k, $k = 1, 2, 3$, are given by the structure constants (see Eq. (7.10)). For the commutator we therefore have

$$[t_k, t_l]_{ij} = -\varepsilon_{kim}\varepsilon_{lmj} + \varepsilon_{lim}\varepsilon_{kmj} = \varepsilon_{klm}\varepsilon_{mij} = i\varepsilon_{klm}(-i\varepsilon_{mij}) = i\varepsilon_{klm}(t_m)_{ij} \tag{7.16}$$

where the second identity follows using the Jacobi identity. This means that the matrices t_k fulfill the Lie algebra

$$[t_k, t_l] = i\varepsilon_{klm}t_m.$$

The t_ks form the adjoint representation of $SU(2)$.

7.2 Quark model of hadrons

It is experimentally well established that the proton and the neutron have inner structure. The evidence is:

- Finite electromagnetic charge radius

  $$\langle r_{p,n}\rangle = 0.8 \cdot 10^{-15}\,\mathrm{m}$$

  (The neutron is to be thought of as a neutral cloud of electromagnetically interacting constituents.)

- Anomalous magnetic moment

  $$\vec{\mu} = g\frac{q}{2m}\vec{s} \qquad g_p = 5.59 \qquad g_n = -3.83$$

- Proliferation of strongly interacting hadronic states (particle zoo)

  $$p,\ n,\ \Lambda,\ \Delta^-,\ \Xi,\ \Sigma,\ \Omega,\ \ldots$$

The explanation for these phenomena is that protons and neutrons (and the other hadrons) are bound states of quarks:

$$\left.\begin{array}{l} |p\rangle = |uud\rangle \\ |n\rangle = |udd\rangle \end{array}\right\}\ \text{3 quark states.}$$

The up quark and the down quark have the following properties

$$|u\rangle:\ q = +\frac{2}{3},\ I = \frac{1}{2},\ I_3 = +\frac{1}{2},\ S = \frac{1}{2};$$
$$|d\rangle:\ q = -\frac{1}{3},\ I = \frac{1}{2},\ I_3 = -\frac{1}{2},\ S = \frac{1}{2}.$$

Quarks			Charge	Baryon number
Up 1.5 − 3 Mev	Charm 1270 MeV	Top 171 000 MeV	+2/3 e	1/3
Down 3.5 − 6 MeV	Strange 105 MeV	Bottom 4200 MeV	−1/3 e	1/3
Leptons			Charge	Lepton number
e^-	μ^-	τ^-	− e	1
ν_e	ν_μ	ν_τ	0	1

Table 7.1: Quarks and leptons.

Thus, $|u\rangle$ and $|d\rangle$ form an isospin doublet and combining them yields the correct quantum numbers for $|p\rangle$ and $|n\rangle$. There are also quark-antiquark bound states: The pions form an isospin triplet while the $|\eta\rangle$ is the corresponding singlet state (see Eq. (7.11)):

$$\left.\begin{aligned} |\pi^+\rangle &= |u\bar{d}\rangle \\ |\pi^0\rangle &= \tfrac{1}{\sqrt{2}}\left(|u\bar{u}\rangle - |d\bar{d}\rangle\right) \\ |\pi^-\rangle &= |d\bar{u}\rangle \end{aligned}\right\} \text{triplet states, } I = 1$$

$$\left. |\eta\rangle = \tfrac{1}{\sqrt{2}}\left(|u\bar{u}\rangle + |d\bar{d}\rangle\right)\right\} \text{singlet state, } I = 0.$$

There are in total three known quark doublets:

$$\underbrace{\begin{pmatrix} |u\rangle \\ |d\rangle \end{pmatrix}}_{\text{up/down}} \qquad \underbrace{\begin{pmatrix} |c\rangle \\ |s\rangle \end{pmatrix}}_{\text{charm/strange}} \qquad \underbrace{\begin{pmatrix} |t\rangle \\ |b\rangle \end{pmatrix}}_{\text{top/bottom}} \qquad \begin{pmatrix} q = +\frac{2}{3},\ I_3 = +\frac{1}{2} \\ q = -\frac{1}{3},\ I_3 = -\frac{1}{2} \end{pmatrix}.$$

These quarks can be combined to give states like, e. g., $|\Lambda\rangle = |uds\rangle$.

7.3 Hadron spectroscopy

7.3.1 Quarks and leptons

Experimental evidence shows that, in addition to the three quark isospin doublets, there are also three families of leptons, the second type of elementary fermions (see Tab. 7.1). The lepton families are built out of an electron (or μ or τ) and the corresponding neutrino. The summary also shows the large mass differences between the six known quarks. All of the listed particles have a corresponding antiparticle, carrying opposite charge and baryon or lepton number, respectively.

Stable matter is built out of quarks and leptons listed in the first column of the family table. Until now, there is no evidence for quark substructure and they are therefore considered to be elementary. Hadrons, on the other hand, are composite particles. They are divided in two main categories as shown in the following table:

Quarks	Flavor	Other numbers
Up, Down	—	$S = C = B = T = 0$
Charm	$C = +1$	$S = B = T = 0$
Strange	$S = -1$	$C = B = T = 0$
Top	$T = +1$	$S = C = B = 0$
Bottom	$B = -1$	$S = C = T = 0$

Table 7.2: Additional quantum numbers for the characterization of unstable hadronic matter. Antiquarks have opposite values for these quantum numbers.

Type	Matter	Antimatter
Baryons	qqq	$\bar{q}\bar{q}\bar{q}$
Mesons	$q\bar{q}$	

Bound states such as $|qq\rangle$ or $|qq\bar{q}\rangle$ are excluded by the theory of quantum chromodynamics (see Sect. 7.4).

Unstable hadronic matter is characterized by the following additional flavor quantum numbers: Charm (C), Strangeness (S), Beauty (B), and Topness (T) (see Tab. 7.2). It is important to remember that in strong and electromagnetic interactions both baryon and flavor quantum numbers are conserved while in weak interactions only baryon quantum numbers are conserved. Therefore, weak interactions allow heavy quarks to decay into the stable quark family. The quark decay channels are shown in the following table:

Quark $\to$	Decay products
$u,\ d$	stable
s	uW^-
c	sW^+
b	cW^-
t	bW^+

As we have seen, protons and neutrons are prominent examples of baryons. Their general properties can be summarized as follows:

	Proton	**Neutron**
Quarks	$\|uud\rangle$	$\|udd\rangle$
Mass	0.9383 GeV	0.9396 GeV
Spin	1/2	1/2
Charge	$\mathrm{e} = 1.6 \cdot 10^{-19}$ C	0 C
Baryon number	1	1
Lifetime	stable: $\tau \geq 10^{32}$ years	unstable: $\tau_{n \to pe^- \nu_e} = 887 \pm 2$ s
Production	gaseous hydrogen: ionization through electric field	under 1 MeV: nuclear reactors; $1 - 10$ MeV: nuclear reactions
Target for experiments	liquid hydrogen	liquid deuterium

The respective antiparticles can be produced in high-energy collisions, e. g.

$$pp \to ppp\bar{p} \;\text{ with }\; |\bar{p}\rangle = \left|\bar{u}\bar{u}\bar{d}\right\rangle \;\text{ or}$$
$$pp \to ppn\bar{n} \;\text{ with }\; |\bar{n}\rangle = \left|\bar{u}\bar{d}\bar{d}\right\rangle .$$

Recall that in Sect. 4.1 we calculate the energy threshold for the reaction $pp \to ppp\bar{p}$ and find that a proton beam colliding against a proton target must have at least $|\vec{p}| = 6.5\,\text{GeV}$ for the reaction to take place.

7.3.2 Strangeness

We now take a more detailed look at the strangeness quantum number. In 1947, a new neutral particle, K^0, was discovered from interactions of cosmic rays:

$$\pi^- p \overset{\text{s}}{\to} K^0\Lambda, \text{ with consequent decays: } K^0 \overset{\text{w}}{\to} \pi^+\pi^-,\ \Lambda \overset{\text{w}}{\to} \pi^- p. \qquad (7.17)$$

This discovery was later confirmed in accelerator experiments. The processes in Eq. (7.17) is puzzling because the *production* cross section is characterized by the strong interaction while the long lifetime ($\tau \sim 90\,\text{ps}$) indicates a weak *decay*. In this seemingly paradoxical situation, a new quantum number called "strangeness" is introduced. A sketch of production and decay of the K^0 is shown in Fig. 7.2. As stated before, the strong interaction conserves flavor which requires for the production $\Delta S = 0$. The decay, on the other hand, proceeds through the weak interaction: The s-quark decays via $s \to uW^-$.

Baryons containing one or more strange quarks are called hyperons. With three constituting quarks we can have, depending on the spin alignment, spin-1/2 ($|\uparrow\downarrow\uparrow\rangle$) or spin-3/2 ($|\uparrow\uparrow\uparrow\rangle$) baryons (see Tab. 7.3).[2] There are 8 spin-1/2 baryons (octet) and 10 spin-3/2 baryons (decuplet). Octet and decuplet are part of the $SU(3)$ multiplet structure (see Sect. 7.4).[3] All hyperons in the octet decay weakly (except for the Σ^0). They therefore have a long lifetime of about $10^{-10}\,\text{s}$ and decay with $|\Delta S| = 1$, e. g.

$$\Sigma^+ \to p\pi^0,\ n\pi^+$$
$$\Xi^0 \to \Lambda\pi^0.$$

The members of the decuplet, on the other hand, all decay strongly (except for the Ω^-) with $|\Delta S| = 0$. They therefore have short lifetimes of about $10^{-24}\,\text{s}$, e. g.

$$\Delta^{++}(1230) \to \pi^+ p$$
$$\Sigma^+(1383) \to \Lambda\pi^+.$$

[2] The problem that putting three fermions into one symmetric state violates the Pauli exclusion principle is discussed in Sect. 7.4.

[3] However, this "flavor $SU(3)$" is only a sorting symmetry and has nothing to do with "color $SU(3)$" discussed in Sect. 7.4.

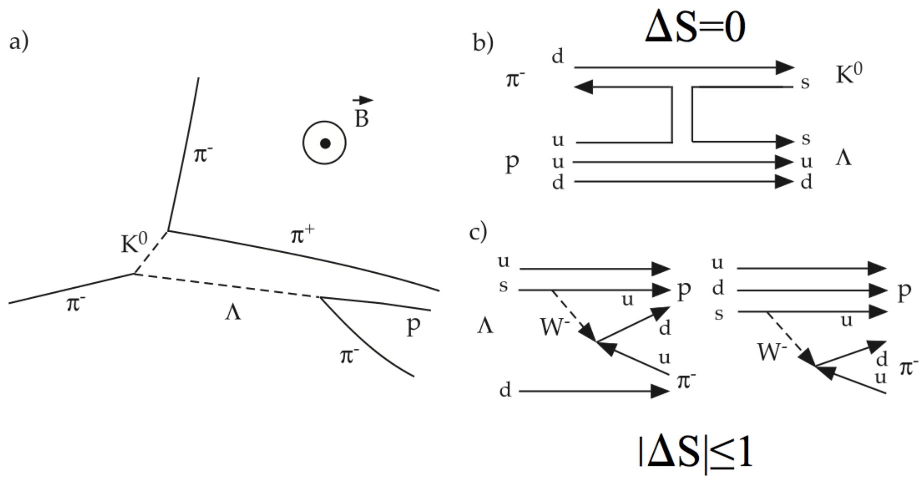


Figure 7.2: *Sketch of the reaction* $\pi^- p \to K^0 \Lambda$ *and the decays of the neutral* K^0 *and* Λ. Tracks detected in a bubble chamber (a). Feynman diagrams for the production and the Λ decay (b). Notice that $S(K^0) = 1,\ |K^0\rangle = |d\bar{s}\rangle$ and $S(\Lambda) = -1,\ |\Lambda\rangle = |uds\rangle$. Source: [8, p. 140].

Spin-1/2: **Octet**			Spin-3/2: **Decuplet**		
Baryon	State	Strangeness	Baryon	State	Strangeness
$p(938)$	$\|uud\rangle$	0	$\Delta^{++}(1230)$	$\|uuu\rangle$	0
$n(940)$	$\|udd\rangle$	0	$\Delta^{+}(1231)$	$\|uud\rangle$	0
$\Lambda(1115)$	$\|(ud-du)s\rangle$	-1	$\Delta^{0}(1232)$	$\|udd\rangle$	0
$\Sigma^{+}(1189)$	$\|uus\rangle$	-1	$\Delta^{-}(1233)$	$\|ddd\rangle$	0
$\Sigma^{0}(1192)$	$\|(ud+du)s\rangle$	-1	$\Sigma^{+}(1383)$	$\|uus\rangle$	-1
$\Sigma^{-}(1197)$	$\|dds\rangle$	-1	$\Sigma^{0}(1384)$	$\|uds\rangle$	-1
$\Xi^{0}(1315)$	$\|uss\rangle$	-2	$\Sigma^{-}(1387)$	$\|dds\rangle$	-1
$\Xi^{-}(1321)$	$\|dss\rangle$	-2	$\Xi^{0}(1532)$	$\|uss\rangle$	-2
			$\Xi^{-}(1535)$	$\|dss\rangle$	-2
			$\Omega^{-}(1672)$	$\|sss\rangle$	-3

Table 7.3: Summary of the baryon octet and decuplet.

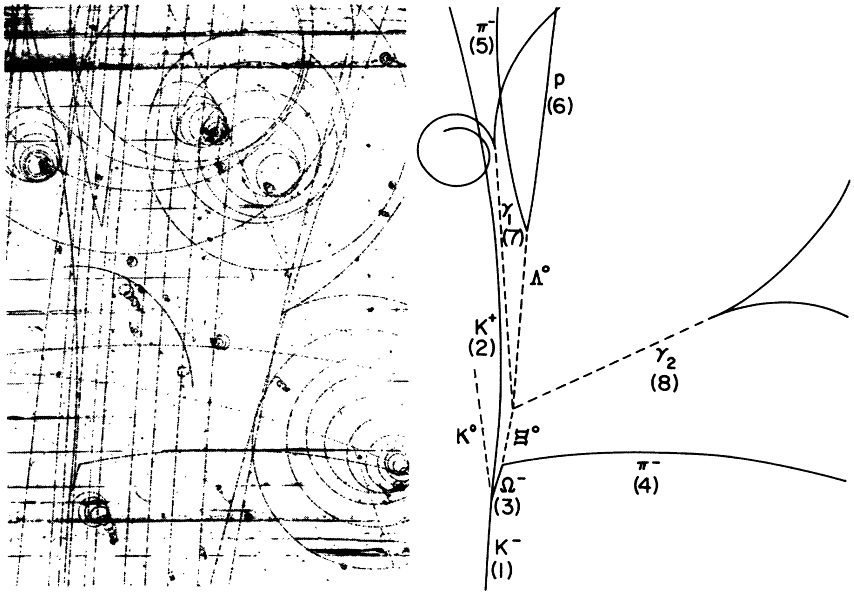


Figure 7.3: *Bubble chamber photograph (LHS) and line diagram (RHS) of an event showing the production and decay of* Ω^-. Source: [25, p. 205].

The quark model, as outlined so far, predicts the hyperon $|\Omega^-\rangle = |sss\rangle$ as a member of the spin-3/2 decuplet. Therefore, the observation of the production,

$$K^- p \to \Omega^- K^+ K^0,$$

and decay,

$$\Omega^- \to \Xi^0 \pi^-, \; \Xi^0 \to \Lambda \pi^0, \; \Lambda \to p \pi^-,$$

of the Ω^- at Brookhaven in 1964 is a remarkable success for the quark model. A sketch of the processes is given in Fig. 7.3. Note that the production occurs via a strong process, $\Delta S = 0$, while the decay is weak: $|\Delta S| = 1$.

7.3.3 Strong vs. weak decays

Generally speaking, strong processes yield considerably shorter lifetimes than weak processes. Consider, for instance, the following two decays,

$\Delta^+ \to p + \pi^0$	$\Sigma^+ \to p + \pi^0$
$\tau_\Delta = 6 \cdot 10^{-24}\,\mathrm{s}$	$\tau_\Sigma = 8 \cdot 10^{-11}\,\mathrm{s}$
$|uud\rangle \to |uud\rangle + \frac{1}{\sqrt{2}}\left(|u\bar{u}\rangle + |d\bar{d}\rangle\right)$	$|uus\rangle \to |uud\rangle + \frac{1}{\sqrt{2}}\left(|u\bar{u}\rangle + |d\bar{d}\rangle\right)$
(strong)	(weak).

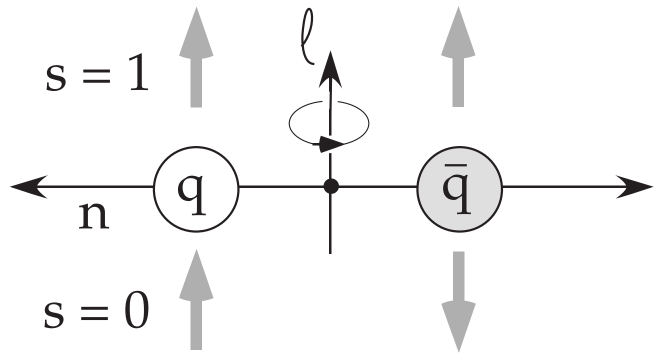


Figure 7.4: *Sketch of the possible spin configurations for quark-antiquark bound states.* The $q\bar{q}$ pair is characterized by orbital excitations l (rotation) and radial excitations n (vibration). Source: [8, p. 141].

The final state is identical in both decays but the lifetime is much longer for the weak process. Since the final state is equal, this difference in lifetime must come from a difference in the coupling constants. For $\tau \sim 1/\alpha^2$ where α is a coupling constant:

$$\frac{\alpha_{\text{weak}}}{\alpha_{\text{strong}}} \sim \sqrt{\frac{\tau_\Delta}{\tau_\Sigma}} = 2.7 \cdot 10^{-7}.$$

7.3.4 Mesons

Mesons are quark-antiquark bound states: $|q\bar{q}\rangle$. In analogy to the spin states of a two-electron system (and not to be confused with the isospin multiplets discussed on p. 128), the $|q\bar{q}\rangle$ bound state can have either spin 0 (singlet) or spin 1 (triplet) (see Fig. 7.4). Radial vibrations are characterized by the quantum number n while orbital angular momentum is characterized by the quantum number l. The states are represented in spectroscopic notation:

$$n^{2s+1}l_J$$

where $l = 0$ is labeled by S, $l = 1$ by P and so on. A summary of the $n = 1$, $l = 0$ meson states is shown in Tab. 7.4. A summary of the states with $l \leq 2$ can be found in Fig. 7.3.4.

7.3.5 Gell-Mann-Nishijima formula

Isospin is introduced in Sect. 7.1. The hadron isospin multiplets for $n = 1$, $l = 0$ are shown in Fig. 7.6. This summary leads to the conclusion that the charge Q of an hadron with

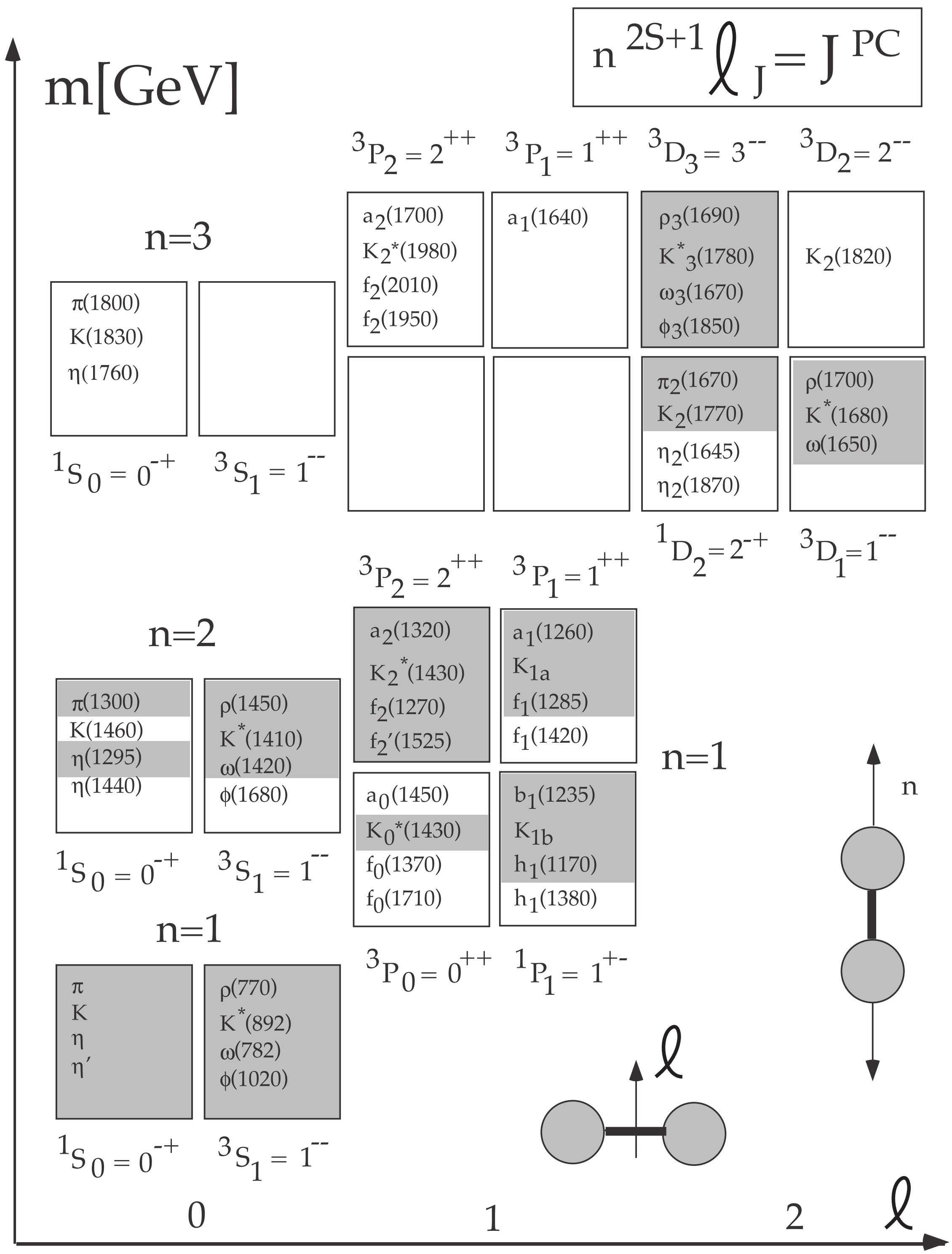


Figure 7.5: *Summary of mesons from u, d s quarks for $l \leq 2$.* Cells shaded in grey are well established states. Source: [8, p. 143].

Mesons ($n = 1,\ l = 0$)			
1^1S_0 (spin 0)		1^3S_1 (spin 1)	
$\pi^+(140)$	$\lvert u\bar{d}\rangle$	$\rho^+(770)$	$\lvert u\bar{d}\rangle$
$\pi^-(140)$	$\lvert \bar{u}d\rangle$	$\rho^-(770)$	$\lvert \bar{u}d\rangle$
$\pi^0(135)$	$1/\sqrt{2}\,\lvert d\bar{d} - u\bar{u}\rangle$	$\rho^0(770)$	$1/\sqrt{2}\,\lvert d\bar{d} - u\bar{u}\rangle$
$K^+(494)$	$\lvert u\bar{s}\rangle$	$K^{*+}(892)$	$\lvert u\bar{s}\rangle$
$K^-(494)$	$\lvert \bar{u}s\rangle$	$K^{*-}(892)$	$\lvert \bar{u}s\rangle$
$K^0(498)$	$\lvert d\bar{s}\rangle$	$K^{*0}(896)$	$\lvert d\bar{s}\rangle$
$\bar{K}^0(498)$	$\lvert \bar{d}s\rangle$	$\bar{K}^{*0}(896)$	$\lvert \bar{d}s\rangle$
$\eta(547)$	$\sim 1/\sqrt{6}\,\lvert u\bar{u} + d\bar{d} - 2s\bar{s}\rangle$	$\phi(1020) = \psi_1$	$-\lvert s\bar{s}\rangle$
$\eta'(958)$	$\sim 1/\sqrt{3}\,\lvert u\bar{u} + d\bar{d} + s\bar{s}\rangle$	$\omega(782) = \psi_2$	$1/\sqrt{2}\,\lvert u\bar{u} + d\bar{d}\rangle$

Table 7.4: Summary of $n = 1$, $l = 0$ meson states.

baryon number B and strangeness S is given by

$$\boxed{Q = I_3 + \frac{B+S}{2}}$$

which is called Gell-Mann-Nishijima formula. As an example, consider the Ω^- hyperon where $0 + (1-3)/2 = -1$.

7.4 Quantum chromodynamics and color $SU(3)$

The quark model, as discussed so far, runs into a serious problem: Since the quarks have half-integer spin, they are fermions and therefore obey Fermi-Dirac statistics. This means that states like

$$\Delta^{++} = \left|u^\uparrow u^\uparrow u^\uparrow\right\rangle,\ S = \frac{3}{2}$$

where three quarks are in a symmetric state (have identical quantum numbers) are forbidden by the Pauli exclusion principle.

The way out is to introduce a new quantum number that allows for one extra degree of freedom which enables us to antisymmetrize the wave function as required for fermions:

$$\Delta^{++} = \mathcal{N} \sum_{ijk} \varepsilon_{ijk} \left|u_i^\uparrow u_j^\uparrow u_k^\uparrow\right\rangle$$

where $\mathcal{N}$ is some normalization constant and the quarks come in three different "colors":[4]

$$|q\rangle \to |q_{1,2,3}\rangle = \begin{pmatrix} |q_1\rangle \\ |q_2\rangle \\ |q_3\rangle \end{pmatrix}.$$

[4]The new charge is named "color" because of the similarities to optics: There are three fundamental colors, complementary colors and the usual combinations are perceived as white.

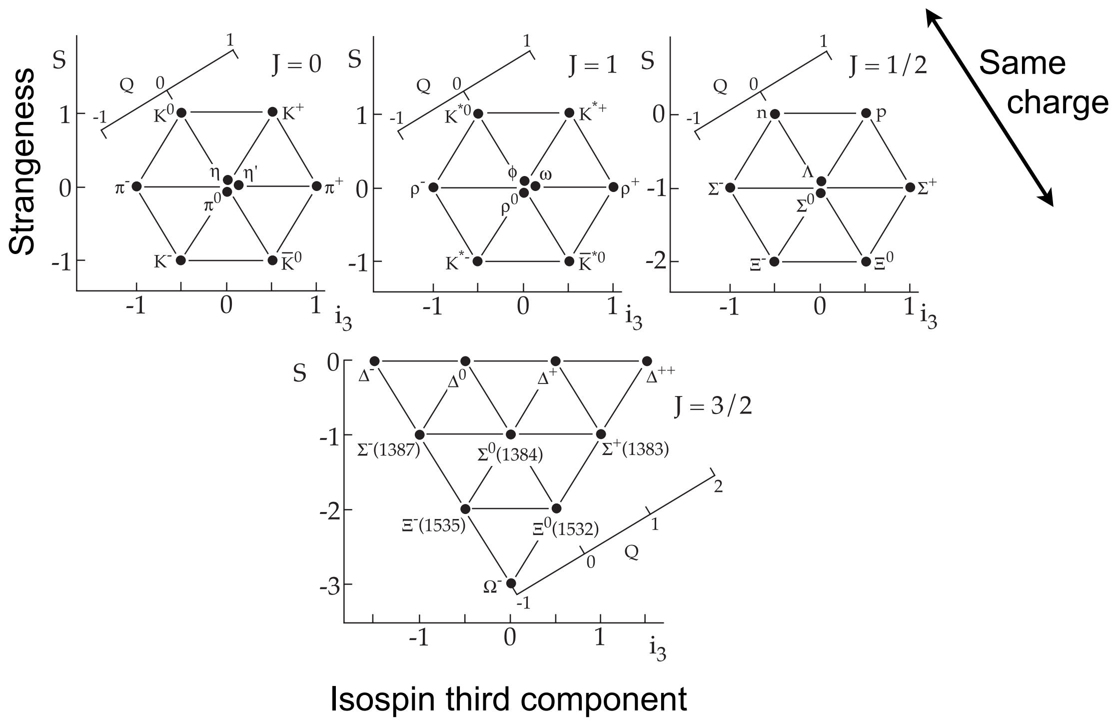


Figure 7.6: *Summary of hadron isospin multiplets.* $n = 1$, $l = 0$. Source: [8, p. 147].

Since color cannot be observed, there has to be a corresponding new symmetry in the Lagrangian due to the fact that the colors can be transformed without the observables being affected. In the case of our new charge in three colors the symmetry group is $SU(3)$, the group of the special unitary transformations in three dimensions. The Lie algebra of $SU(3)$ is

$$\left[T^a, T^b\right] = i f^{abc} T^c$$

where, in analogy to Eq. (7.10), f^{abc} denotes the structure constants and where there are 8 generators T^a (recall that $o = N^2 - 1 = 8$, see p. 127) out of which $r = N - 1 = 2$ are diagonal.

The fundamental representation is given by the 3×3 matrices $T^a = \frac{1}{2}\lambda^a$ with the Gell-Mann matrices

$$\lambda_1 = \begin{pmatrix} \overbrace{\boxed{\begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix}}}^{\tau_1} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & 0 \end{pmatrix} \quad \lambda_2 = \begin{pmatrix} \overbrace{\boxed{\begin{matrix} 0 & -i \\ i & 0 \end{matrix}}}^{\tau_2} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & 0 \end{pmatrix} \quad \lambda_3 = \begin{pmatrix} \overbrace{\boxed{\begin{matrix} 1 & 0 \\ 0 & -1 \end{matrix}}}^{\tau_3} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & 0 \end{pmatrix} \quad \lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$\lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \quad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad \lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \quad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

One can observe that these matrices are hermitian and traceless,

$$\lambda_a^\dagger = \lambda_a \qquad\qquad \mathrm{Tr}\,\lambda^a = 0.$$

Furthermore, one can show that

$$\mathrm{Tr}\left(\lambda^a \lambda^b\right) = 2\delta^{ab}$$

and

$$\lambda^a_{ij}\lambda^b_{kl} = 2\left(\delta_{il}\delta_{kj} - \frac{1}{3}\delta_{ij}\delta_{kl}\right) \text{ (Fierz identity).}$$

The structure constants of $SU(3)$ are given by

$$f_{abc} = \frac{1}{4i}\mathrm{Tr}\left(\left[\lambda_a, \lambda_b\right]\lambda_c\right)$$

and are antisymmetric in a, b, and c. The numerical values are

$$\begin{aligned} &f_{123} = 1 \\ &f_{458} = f_{678} = \frac{\sqrt{3}}{2} \\ &f_{147} = f_{156} = f_{246} = f_{257} = f_{345} = f_{367} = \frac{1}{2} \\ &f_{abc} = 0 \quad \text{else.} \end{aligned}$$

As in the case of $SU(2)$, the adjoint representation is given by the structure constants which, in this case, are 8×8 matrices:

$$(t^a)_{bc} = -i f_{abc}.$$

The multiplets (again built out of the fundamental representations) are given by the direct sums

$$\underline{3} \otimes \underline{\bar{3}} = \underline{1} \oplus \underline{8} \tag{7.18}$$

where the bar denotes antiparticle states and

$$\underline{3} \otimes \underline{3} \otimes \underline{3} = \underline{1} \oplus \underline{8} \oplus \underline{8} \oplus \underline{10}. \tag{7.19}$$

The singlet in Eq. (7.18) corresponds to the $|q\bar{q}\rangle$ states, the mesons (e. g. π), while the singlet in Eq. (7.19) is the $|qqq\rangle$ baryon (e. g. p, n). The other multiplets are colored and can thus not be observed.'Working out the$SU(3)$ potential structure, one finds that an attractive QCD potential exists only for the singlet states, while the potential is repulsive for all other multiplets.

The development of QCD outlined so far can be summarized as follows: Starting from the observation that the nucleons have similar properties, we considered isospin and $SU(2)$ symmetry. We found that the nucleons n and p correspond to the fundamental representations of $SU(2)$ while the π is given by the adjoint representation. To satisfy the Pauli exclusion principle, we had to introduce a new quantum number and with it a new $SU(3)$ symmetry of the Lagrangian. This in turn led us to multiplet structures where the colorless singlet states correspond to mesons and baryons.

Construction of QCD Lagrangian We now take a closer look at this $SU(3)$ transformation of a color triplet,

$$|q\rangle = \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} \to |q'\rangle = \begin{pmatrix} q'_1 \\ q'_2 \\ q'_3 \end{pmatrix} = \mathrm{e}^{ig_s\alpha_a T^a} \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} = U\,|q\rangle\,, \tag{7.20}$$

where $g_s \in \mathbb{R}$ is used as a rescaling (and will be used for the perturbative expansion) of the group parameter α introduced previously. The reason of introducing it becomes clear in the context of gauge theories.

In analogy to the QED current,

$$j^\mu_{\text{QED}} = eq_e\bar{q}\gamma^\mu q,$$

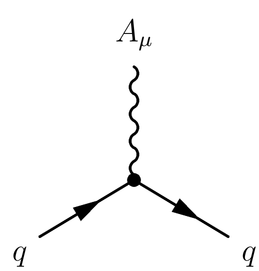
we introduce the **color current** [5], which is the conserved current associated with the $SU(3)$ symmetry,

$$j^\mu_a = g_s\bar{q}_i\gamma^\mu T^a_{ij}q_j \qquad\qquad a = 1\cdots 8. \tag{7.21}$$

In the same spirit, by looking at the QED interaction,

$$\mathcal{L}^{\text{int}}_{\text{QED}} = -j^\mu_{\text{QED}}A_\mu = eq_e\bar{q}\gamma^\mu qA_\mu,$$

yielding the vertex,



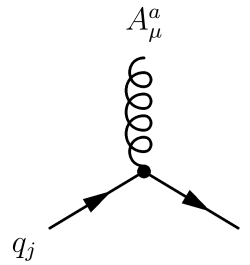
where we can see the photon – the electrically uncharged $U(1)$ gauge boson of QED –, we

[5]The Einstein summation convention still applies, even if the color index i and j are not in an upper and lower position. This exception extends also to the color indices $a, b, ...$ of the gauge fields to be introduced. There is no standard convention in the literature, and since there is no metric tensor involved, the position of a color index, is merely an esthetic/readability problem.

postulate an interaction part of the QCD Lagrangian of the form,

$$\mathcal{L}_{\text{QCD}}^{\text{int}} = -j_a^\mu A_\mu^a = g_s \bar{q}_i \gamma^\mu T_{ij}^a q_j A_\mu^a, \tag{7.22}$$

which translates in the vertex (which is not the only one of QCD as we shall see),



Now there are 8 $SU(3)$ gauge bosons A_μ^a for QCD : one for each possible value of a. They are called **gluons** and are themselves colored.

Continuing with our analogy, we define the **covariant derivative of QCD** [6],

$$D_\mu = \partial_\mu \mathbb{1} + i g_s T^a A_\mu^a, \tag{7.23}$$

and state that the QCD Lagrangian should have a term of the form,

$$\tilde{\mathcal{L}}_{\text{QCD}} = \bar{q}(i\not{D} - m)q. \tag{7.24}$$

Up to this point, both QED and QCD look nearly identical. Their differences become crucial when we look at local gauge symmetries. Such a transformation can be written,

$$|q(x)\rangle \to |q'(x)\rangle = \mathrm{e}^{i g_s \alpha_a(x) T^a} |q(x)\rangle \,, \tag{7.25}$$

and we impose as before that the Lagrangian must be invariant under any such transformation. This is equivalent of imposing,

$$\begin{aligned} D'_\mu |q'(x)\rangle &\overset{!}{=} \mathrm{e}^{i g_s \alpha_a(x) T^a} D_\mu |q(x)\rangle \\ &\Leftrightarrow \langle \bar{q}'(x)| \, i\not{D}' \, |q'(x)\rangle = \langle \bar{q}(x)| \, i\not{D} \, |q(x)\rangle \,. \end{aligned}$$

For $\alpha_a(x) \ll 1$, we can expand the exponential and keep only the first order term,

$$\begin{aligned} D'_\mu |q'(x)\rangle &= \left(\partial_\mu + i g_s T^c A'^c_\mu\right)\left(\mathbb{1} + i g_s \alpha_a(x) T^a\right) |q(x)\rangle \\ &\overset{!}{=} \left(\mathbb{1} + i g_s \alpha_a(x) T^a\right) \underbrace{\left(\partial_\mu + i g_s T^c A^c_\mu\right)}_{D_\mu} |q(x)\rangle \,. \end{aligned}$$

[6] Note that D_μ acts on color triplet and gives back a color triplet; ∂_μ does not mix the colors, whereas the other summand does (T^a is a 3×3 matrix).

Making the ansatz $A'^c_\mu = A^c_\mu + \delta A^c_\mu$ where $|\delta A^c_\mu| \ll |A^c_\mu|$ and expanding the former equation to first order in δA^c_μ (the term proportional to $\alpha_a(x)\delta A^c_\mu$ has also been ignored), we get,

$$\begin{aligned} & i g_s T^c \delta A^c_\mu + i g_s (\partial_\mu \alpha_a(x)) T^a + i^2 g_s^2 T^c A^c_\mu \alpha_a(x) T^a \stackrel{!}{=} i^2 g_s^2 \alpha_a(x) T^a T^c A^c_\mu \\ & \Rightarrow T^c \delta A^c_\mu \stackrel{!}{=} -(\partial_\mu \alpha_a(x)) T^a + i g_s [T^a, T^c] \alpha_a(x) A^c_\mu, \end{aligned}$$

or, renaming the dummy indices and using the Lie algebra $su(3)$,

$$\begin{aligned} T^a \delta A^a_\mu &= -(\partial_\mu \alpha_a(x)) T^a - g_s f_{abc} T^a \alpha_b(x) A^c_\mu \qquad\qquad \forall T^a \\ \Rightarrow A'^a_\mu &= A^a_\mu - \underbrace{\partial_\mu \alpha_a(x)}_{\text{like in QED}} - \underbrace{g_s f_{abc} \alpha_b(x) A^c_\mu}_{\text{non-abelian part}} . \end{aligned} \tag{7.26}$$

Eq. (7.26) describes the (infinitesimal) gauge transformation of the gluon field.

In order for the gluon field to become physical, we need to include a kinematical term (depending on the derivatives of the field). Remember the photon term of QED,

$$\mathcal{L}^{\text{photon}}_{\text{QED}} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \qquad\qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu,$$

where the last is gauge invariant. As we might expect from Eq. (7.26), the non-abelian part will get us into trouble. Let's look at,

$$\begin{aligned} \delta(\partial_\mu A^c_\nu - \partial_\nu A^c_\mu) = -\partial_\mu \partial_\nu \alpha_a + \partial_\nu \partial_\mu \alpha_a - g_s f_{abc} \alpha_b (\partial_\mu A^c_\nu - \partial_\nu A^c_\mu) \\ - g_s f_{abc} \left((\partial_\mu \alpha_b) A^c_\nu - (\partial_\nu \alpha_b) A^c_\mu \right) . \end{aligned}$$

We remark that the two first summands cancel each other and that the third looks like the $SU(3)$ transformation under the adjoint representation.

We recall that,

$$\begin{aligned} q_i \to q'_i &= (\delta_{ij} + i g_s \alpha_a T^a_{ij}) q_j \qquad\qquad \text{(fundamental representation)} \\ B_a \to B'_a &= (\delta_{ac} + i g_s \alpha_b t^b_{ac}) B_c \qquad\qquad \text{(adjoint representation)} \end{aligned}$$

respectively, where,

$$t^b_{ac} = -i f_{bac} = i f_{abc}.$$

Hence, if $F^a_{\mu\nu}$ transforms in the adjoint representation of $SU(3)$, we should have,

$$\delta F^a_{\mu\nu} \stackrel{!}{=} -g_s f_{abc} \alpha_b F^c_{\mu\nu}.$$

We now make the ansatz,

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g_s f_{abc} A^b_\mu A^c_\nu, \tag{7.27}$$

and prove that it fulfills the above constraint.

$$\begin{aligned}\delta F^a_{\mu\nu} =&\delta(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu) - g_s f_{abc}\delta(A^b_\mu A^c_\nu)\\ =&-g_s f_{abc}\alpha_b(\partial_\mu A^c_\nu - \partial_\nu A^c_\mu) - g_s f_{abc}\left((\partial_\mu \alpha_b)A^c_\nu - (\partial_\nu\alpha_b)A^c_\mu\right)\\ &- g_s f_{abc}\left(-(\partial_\mu\alpha_b)A^c_\nu + (\partial_\nu\alpha_b)A^c_\mu\right) - g_s f_{abc}\left(-g_s f_{bde}\alpha_d A^e_\mu A^c_\nu - g_s f_{cde}\alpha_d A^b_\mu A^e_\nu\right),\end{aligned}$$

Using,

$$f_{abc}f_{bde}\alpha_d A^e_\mu A^c_\nu = f_{abe}f_{bdc}\alpha_d A^c_\mu A^e_\nu = f_{ace}f_{cdb}\alpha_d A^b_\mu A^e_\nu,$$

and

$$f_{aec}f_{dbc} - f_{acb}f_{dec} = (iT^a_{ec})(iT^d_{cb}) - (iT^d_{ec})(iT^a_{cb}) = \left[T^a, T^d\right]_{eb} = if_{adc}T^c_{eb},$$

we get the desired result.

We check finally that a kinematic term based on the above definition of $F^a_{\mu\nu}$ is gauge invariant :

$$\delta\left(F^a_{\mu\nu}F^{\mu\nu}_a\right) = 2F^{\mu\nu}_a\delta F^a_{\mu\nu} = -2g_s \underbrace{f_{abc}}_{=-f_{cba}}\alpha_b \underbrace{F^{\mu\nu}_a F^c_{\mu\nu}}_{=F^{\mu\nu}_c F^a_{\mu\nu}} = 0.$$

Finally, we get the full **QCD Lagrangian**,

$$\mathcal{L}_{\text{QCD}} = -\frac{1}{4}F^a_{\mu\nu}F^{\mu\nu}_a + \bar{q}(i\not{D} - m_q)q, \tag{7.28}$$

with $\not{D}$ and $F^a_{\mu\nu}$ definded by Eqs. (7.23) and (7.27) respectively.

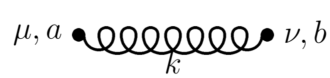
This Lagrangian is per construction invariant under local $SU(3)$ gauge transformations. It is our first example of a non-abelian gauge theory, a so-called **Yang-Mills theory**.

Structure of the kinematic term From the definition of $F^a_{\mu\nu}$, Eq. (7.27), we see that,

$$F^a_{\mu\nu}F^{\mu\nu}_a = \left(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g_s f_{abc}A^b_\mu A^c_\nu\right)\left(\partial^\mu A^\nu_a - \partial^\nu A^\mu_a - g_s f_{ade}A^\mu_d A^\nu_e\right),$$

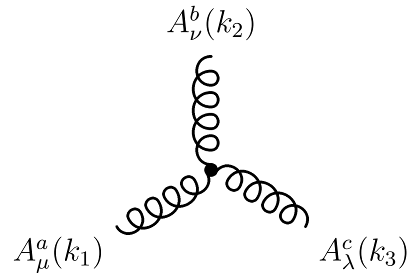
will have a much richer structure than in the case of QED.

First, we have – as in QED – a 2-gluon term $\left(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu\right)\left(\partial^\mu A^\nu_a - \partial^\nu A^\mu_a\right)$ corresponding to the **gluon propagator**,



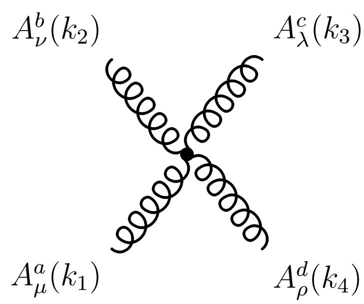
$$= -\frac{g^{\mu\nu}}{k^2}\delta^{ab}. \tag{7.29}$$

Then we have a 3-gluon term $\left(-g_s f_{abc} A^b_\mu A^c_\nu\right)\left(\partial_\mu A^\nu_a - \partial^\nu A^\mu_a\right)$ yielding a **3-gluon vertex**



$$= g_s f_{abc}\left[g_{\mu\nu}(k_1 - k_2)_\lambda + g_{\nu\lambda}(k_2 - k_3)_\mu + g_{\lambda\mu}(k_3 - k_1)_\nu\right]. \tag{7.30}$$

Finally we have also a 4-gluon term $\left(-g_s f_{abc} A^b_\mu A^c_\nu\right)\left(-g_s f_{ade} A^\mu_d A^\nu_e\right)$ yielding the **4-gluon vertex**

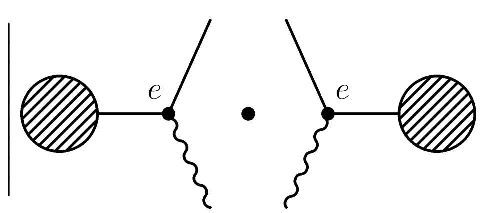


$$= -ig_s^2\left[f_{abe}f_{cde}(g_{\mu\lambda}g_{\nu\rho} - g_{\mu\rho}g_{\nu\lambda}) + f_{ade}f_{bce}(g_{\mu\nu}g_{\lambda\rho} - g_{\mu\lambda}g_{\nu\rho}) + f_{ace}f_{bde}(g_{\mu\rho}g_{\nu\lambda} - g_{\mu\nu}g_{\rho\lambda})\right] \tag{7.31}$$

Unlike in QED, gluons are able to interact with themselves. This comes from the fact that the theory is non-abelian. As a consequence, there is no superposition principle for QCD: the field of a system of strongly interacting particles is *not* the sum of the individual fields. Thence, there is no plane wave solution to QCD problems, and we cannot make use of the usual machinery of Green's functions and Fourier decomposition. Up to now there is no known solution.

7.4.1 Strength of QCD interaction

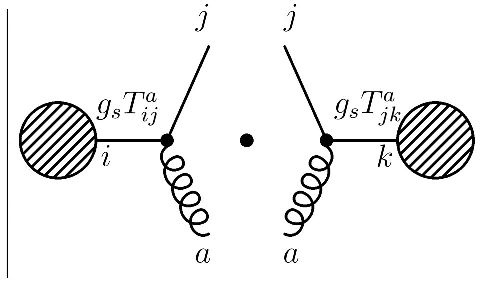
In QED, when we take a term of the form,



where the  denotes some other part of the Feynman diagram, the expression is proportional to $e^2 = 4\pi\alpha$.

In the case of QCD, we have a few more possibilities. We look at the general $SU(n)$ case. The QCD result can be found by setting $n = 3$.

First, for the analogous process to the one cited above :



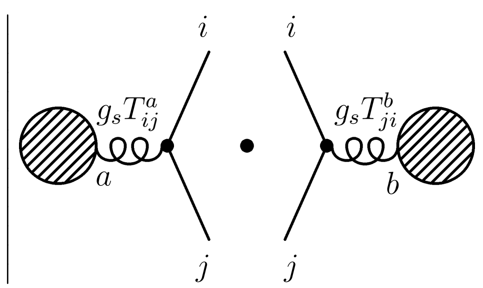
which is proportional to $g_s^2 T^a_{ij} T^a_{jk} = 4\pi\alpha_s C_F \delta_{ik}$, where

$$C_F = \frac{n^2 - 1}{2n}, \tag{7.32}$$

is the color factor, the Casimir operator of $SU(n)$. To find it, we used one of the Fierz identities (see exercises), namely,

$$\begin{aligned} T^a_{ij} T^a_{jk} &= \frac{1}{2}\left(\delta_{ik}\delta_{jj} - \frac{1}{n}\delta_{ij}\delta_{jk}\right) \\ &= \frac{1}{2}\left(n\delta_{ik} - \frac{1}{n}\delta_{ik}\right) = \frac{n^2-1}{2n}\delta_{ik}. \end{aligned}$$

Next we look at,



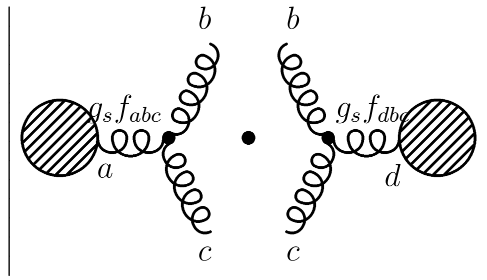
which is proportional to $g_s^2 T^a_{ij} T^b_{ji} = 4\pi\alpha_s T_F \delta^{ab}$, where

$$T_F = \frac{1}{2}. \tag{7.33}$$

To find it, we used the fact that,

$$\mathrm{Tr}\left(T^a T^b\right) = \frac{1}{2}\delta^{ab}.$$

Finally we investigate the case where,



which is proportional to $g_s^2 f_{abc} f_{dbc} = 4\pi\alpha_s C_A \delta^{ad}$, where

$$C_A = n. \tag{7.34}$$

To find it, we used the relation,

$$f_{abc} = -4i\text{Tr}\left(\left[T^a, T^b\right] T^c\right),$$

that we have shown in the beginning of this section.

In the case of QCD, $C_F = \frac{4}{3}, T_F = \frac{1}{2}, C_A = 3$. From the discussion above, we can heuristically draw the conclusion that gluons tend to couple more to other gluons, than to quarks.

At this stage, we note two features specific to the strong interaction, which we are going to handle in more detail in a moment :

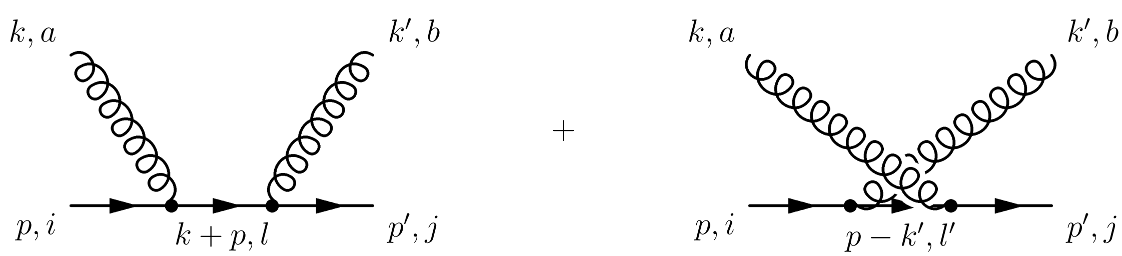
- **Confinement** : At low energies (large distances), the coupling becomes very large, so that the perturbative treatment is no longer valid, an the process of hadronization becomes inportant. This is the reason why we cannot observe color directly.
- **Asymptotic freedom** : At high energies (small distances) the coupling becomes negligible, and the quarks and gluons can move almost freely.

As an example, of typical QCD calculation, we sketch the calculation of the

Gluon Compton scattering

$$g(k) + q(p) \to g(k') + q(p').$$

There are at first sight two Feynman diagrams coming into the calculation,



which yields the following scattering matrix element,

$$-i\mathcal{M}_{fi} = -\, ig_s^2 \left[\bar{u}(p')\not{\varepsilon}^*(k')\frac{1}{\not{p}+\not{k}-m}\not{\varepsilon}(k)u(p)T^b_{jl}T^a_{li} \right.$$
$$\left. +\bar{u}(p')\not{\varepsilon}(k)\frac{1}{\not{p}-\not{k}'-m}\not{\varepsilon}^*(k')u(p)T^b_{jl'}T^a_{l'i}\right] . \tag{7.35}$$

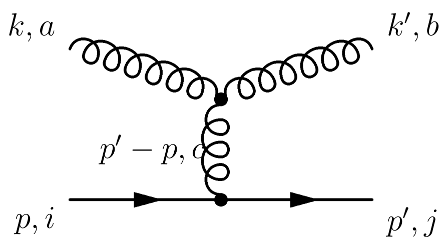
We start by checking the gauge invariance ($\mathcal{M}_{fi}$ must vanish under the substitution $\varepsilon_\mu(k) \to k_\mu$):

$$-i\mathcal{M}'_{fi} = ig_s^2\bar{u}(p')\not{\varepsilon}(k')u(p)\left(T^b_{ji}T^a_{li} - T^a_{jl'}T^b_{l'i}\right),$$

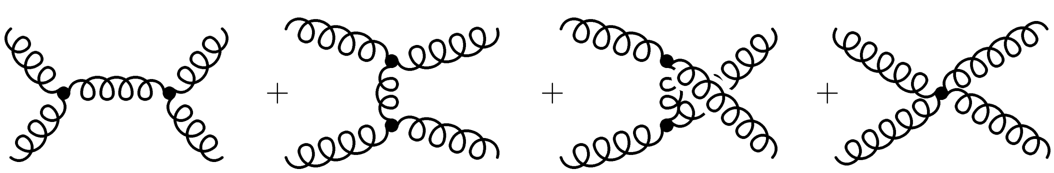
where

$$T^b_{ji}T^a_{li} - T^a_{jl'}T^b_{l'i} = \left[T^b, T^a\right]_{ji} = if_{bac}T^c_{ki} \neq 0!$$

So we need another term, which turns out to be the one corresponding to the Feynman diagram,



The calculation of the gluon-gluon scattering goes analogously. We need to consider the graphs,



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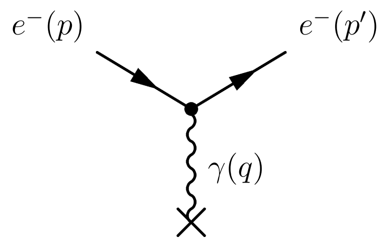
7.4.2 QCD coupling constant

To leading order, a typical QED scattering process takes the form,

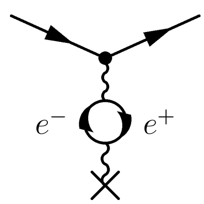


with $q^2 = (p' - p)^2 \leq 0$.

In the Coulomb limit (long distance, low momentum transfer), the potential takes the form,

$$V(R) = -\frac{\alpha}{R} \qquad\qquad R \gtrsim \frac{1}{m_e} \approx 10^{-11}\,[\text{cm}]. \tag{7.36}$$

When $R \leq m_e^{-1}$, quantum effects become important (loop corrections, also known as vacuum polarization), since the next to leading order (NLO) diagram,



starts to play a significant (measurable) role. This results in a change of the potential to,

$$V(R) = -\frac{\alpha}{R}\left[1 + \frac{2\alpha}{3\pi}\ln\frac{1}{m_e R} + \mathcal{O}(\alpha^2)\right] = -\frac{\bar{\alpha}(R)}{R}, \tag{7.37}$$

where $\bar{\alpha}(R)$ is called the effective coupling.

We can understand the effective coupling in analogy to a solid state physics example : in an insulator, an excess of charge gets screened by the polarization of the nearby atoms. Here we create e^+e^- pairs out of the vacuum, hence the name **vacuum polarization**.

As we can see from Eq. (7.37), the smaller the distance $R \leq m_e^{-1}$, the bigger the observed "charge" $\bar{\alpha}(R)$. What we call the electron charge e (or the fine structure constant α) is the limiting value for very large distances or low momentum transfer as shown in Fig. 7.4.2.

For example the measurements done at LEP show that, $\bar{\alpha}(Q^2 = m_Z^2) \approx \frac{1}{128} > \alpha$.

In the case of QCD, we have at NLO, the following diagrams,

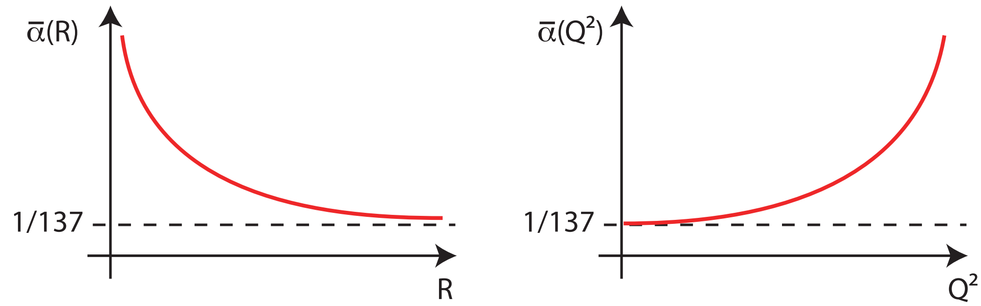
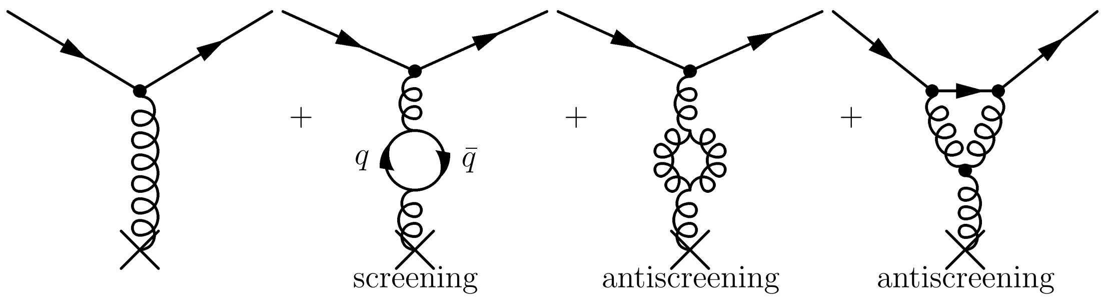


Figure 7.7: Evolution of the effective electromagnetic coupling with distance and energy ($Q^2 = -q^2$).



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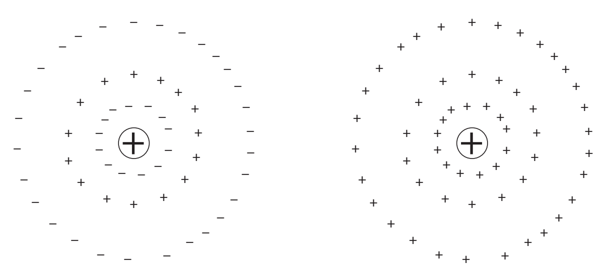
We can picture the screening/antiscreening phenomenon as follows,



$R \searrow \Rightarrow \alpha_s(R) \nearrow$ $\quad$ $R \searrow \Rightarrow \alpha_s(R) \searrow$

Figure 7.8: Screening and antiscreening.

For QCD, the smaller the distance R (or the bigger the energy Q^2), the smaller the observed coupling $\bar{\alpha}_s(R)$. At large distances, $\bar{\alpha}_s(R)$ becomes comparable with unity, and the perturbative approach breaks down as we can see in Fig. 7.4.2. The region concerning confinement and asymptotic freedom are also shown.

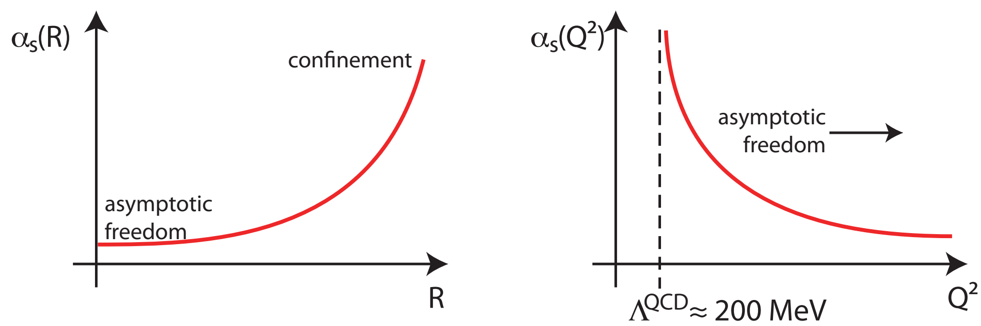


Figure 7.9: Evolution of the effective strong coupling with distance and energy ($Q^2 = -q^2$).

The β-function of QCD In the renormalization procedure of QCD, we get a differential equation for $\alpha_s(\mu^2)$ where μ is the renormalization scale,

$$\mu^2 \frac{\partial \alpha_s}{\partial(\mu^2)} = \beta(\alpha_s) \tag{7.38}$$

$$\beta(\alpha_s) = -\alpha_s \left[\beta_0 \frac{\alpha_s}{4\pi} + \beta_1 \left(\frac{\alpha_s}{4\pi}\right)^2 + \beta_2 \left(\frac{\alpha_s}{4\pi}\right)^3 + \cdots\right], \tag{7.39}$$

with

$$\beta_0 = \frac{11}{3} n_c - \frac{2}{3} n_f = 11 - \frac{2}{3} n_f \qquad \text{(NLO)} \tag{7.40}$$

$$\beta_1 = \frac{17}{12} n_c^2 - \frac{5}{12} n_c n_f - \frac{1}{4}\left(\frac{n_c^2 - 1}{2n_c}\right) n_f, \qquad \text{(NNLO)} \tag{7.41}$$

where n_c is the number of colors and n_f is the number of quark flavors. These two numbers enter into the calculation through gluon respectively quark loop corrections to the propagators.

We remark at this stage that unless $n_f \geq 17$[7], $\beta_0 > 0$, whereas in the case of QED, we get,

$$\beta_0^{\text{QED}} = -\frac{4}{3} < 0. \tag{7.42}$$

This fact explains the completely different behavior of the effective couplings of QCD and QED.

To end this chapter, we will solve Eq. 7.39 retaining only the first term of the power

[7] As of 2009, only 6 quark flavors are known and there is experimental evidence (decay witdth of the Z^0 boson) that there are no more than 3 generations with light neutrinos.

expansion of β.

$$\begin{aligned}
\mu^2 \frac{\partial \alpha_s}{\partial(\mu^2)} &= -\frac{\beta_0}{4\pi}\alpha_s^2 \\
\frac{\partial \alpha_s}{\alpha_s^2} &= -\frac{\beta_0}{4\pi}\partial(\ln\mu^2) \\
\int_{\alpha_s(Q_0^2)}^{\alpha_s(Q^2)} \frac{d\alpha_s}{\alpha_s^2} &= -\frac{\beta_0}{4\pi}\int_{\ln Q_0^2}^{\ln Q^2} d(\ln\mu^2),
\end{aligned}$$

and hence,

$$\boxed{\frac{1}{\alpha_s(Q^2)} = \frac{1}{\alpha_s(Q_0^2)} + \frac{\beta_0}{4\pi}\ln\frac{Q^2}{Q_0^2}}. \tag{7.43}$$

We thus have a relation between $\alpha_s(Q^2)$ and $\alpha_s(Q_0^2)$, giving the evolution of the effective coupling.

A mass scale is also generated, if we set,

$$\frac{1}{\alpha_s(Q^2 = \Lambda^2)} = 0 \Rightarrow \alpha_s(\Lambda^2) = \infty.$$

Choosing $\Lambda = Q_0$, we can rewrite Eq. (7.43) as,

$$\boxed{\alpha_s(Q^2) = \frac{4\pi}{\beta_0 \ln\frac{Q^2}{\Lambda^2}}}. \tag{7.44}$$