Chapter 2

Relativistic kinematics

Literature:

- Nachtmann [4]
- Hagedorn [5]
- Byckling/Kajantie [6]

We state some notation concerning special relativity:

$$
\begin{aligned}
x^\mu &= (x^0 = t, x^1, x^2, x^3) = (t, \vec{x}) && \text{contravariant four-vector} && (2.1)\\
x_\mu &= (t, -\vec{x}) && \text{covariant four-vector} && (2.2)\\
g^{\mu\nu} = g_{\mu\nu} &= \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} && \text{metric tensor} && (2.3)\\
\tau^2 = t^2 - x^2 &= g_{\mu\nu} x^\mu x^\nu = x^\mu x_\mu = x^2 && \text{Lorentz invariant} && (2.4)\\
d\tau &= dt\sqrt{1 - \left(\frac{d\vec{x}}{dt}\right)^2} = \frac{dt}{\gamma} && \text{proper time.} && (2.5)
\end{aligned}
$$

Combining Eq. (2.1) and (2.5) we arrive at the four-velocity

$$u^\mu = \frac{dx^\mu}{d\tau} = \frac{dx^\mu}{dt}\frac{dt}{d\tau} = \gamma(1, \vec{v}).$$

Since

$$\Rightarrow u^2 = \gamma^2(1 - \vec{v}^2) = 1 > 0,$$

u is a time-like four-vector. The four-momentum is then defined as

$$p^\mu = mu^\mu = m\gamma(1, \vec{v}) = (p^0 = E, \vec{p}).$$

By calculating the corresponding Lorentz invariant,

$$p^2 = m^2 u^2 = m^2 = E^2 - \vec{p}^2,$$

we find the energy-momentum relation

$$E = \sqrt{m^2 + \vec{p}^2}. \tag{2.6}$$

A particle is said to be relativistic if $\vec{p}^2 \not\ll m^2$. Conversely, for a non-relativistic particle, $\vec{p}^2 \ll m^2$, and therefore

$$E = \sqrt{m^2 + \vec{p}^2} = m\left(1 + \frac{1}{2}\frac{\vec{p}^2}{m^2} + \ldots\right) = m + \frac{1}{2}\frac{\vec{p}^2}{m} + \ldots$$

so that we recover Newton's expression for $|\vec{v}| \ll 1$.

2.1 Particle decay

The decaying particle's four-momentum is, in the rest frame, given by $p = (M, 0, 0, 0)$, see Fig. 2.1. The decay time (lifetime) is

$$d\tau^2 = dt^2(1 - \vec{v}^2)$$

where dt^2 is the lifetime in the laboratory frame:

$$dt = \gamma d\tau > d\tau. \tag{2.7}$$

The result stated in equation (2.7) has been verified experimentally:

$$\begin{aligned}
&\tau_{\pi^+ \to \mu^+ \nu_\mu} = 2.6 \cdot 10^{-8}\,\mathrm{s} \\
&E_\pi = 20\,\mathrm{GeV},\ \gamma = \frac{E_\pi}{m_\pi} = 143 \Leftrightarrow v = 0.9999 \\
&\Rightarrow \frac{t'_\pi}{t_\pi} = 143.
\end{aligned}$$

Constraints are (i) conservation of energy and momentum, $p = p_1 + p_2$ (4 equations), and (ii) the mass-shell condition:

$$\begin{array}{lll}
p^2 = M^2 & p_1^2 = m_1^2 & p_2^2 = m_2^2 \\
p = (M, \vec{0}) & p_1 = (E_1, \vec{p}_1) & p_2 = (E_2, \vec{p}_2).
\end{array}$$

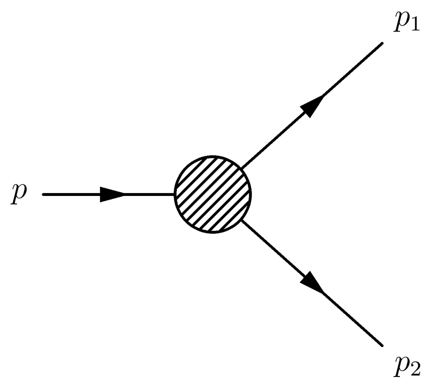

Figure 2.1: *Particle decay.* Dynamics will be discussed later on; at the moment we are dealing with kinematics.

It therefore follows that

$$p \cdot p_i = ME_i \ \Rightarrow E_i = \frac{1}{M} p \cdot p_i = \frac{1}{M}(p_1 \cdot p_i + p_2 \cdot p_i).$$

And, by using $p_1 \cdot p_2 = \frac{1}{2}[(p_1 + p_2)^2 - p_1^2 - p_2^2] = \frac{1}{2}[M^2 - m_1^2 - m_2^2]$, we find

$$\begin{aligned} E_1 &= \frac{1}{M}(p_1^2 + p_1 \cdot p_2) = \frac{1}{2M}(M^2 + m_1^2 - m_2^2) \\ E_2 &= \frac{1}{2M}(M^2 - m_1^2 + m_2^2). \end{aligned}$$

By using equation (2.6) and $\vec{p}_1 + \vec{p}_2 = 0$, the absolute value of the three-momenta,

$$\vec{p}_1^{\,2} = E_1^2 - m_1^2 = \frac{1}{4M^2}\Big(M^4 - 2M^2(m_1^2 + m_2^2) + (m_1^2 - m_2^2)^2\Big) = \vec{p}_2^{\,2},$$

is also fixed. This means that only the directions of $\vec{p}_1$ and $\vec{p}_2$ remain unknown, while the energies and the absolute values of the momenta can be calculated directly.

2.2 Two-particle scattering

For a visualisation of the process see Fig. 2.2(a). Once again, the constraints are

$$\begin{aligned} &p_i^2 = m_i^2 \ (i = 1, \ldots, 4) \\ &p_1 + p_2 = p_3 + p_4. \end{aligned}$$

If $m_1 = m_3$ and $m_2 = m_4$, elastic scattering takes place. Consider the Lorentz invariants

$$p_i^2 = m_i^2 \text{ and } \underbrace{p_1 \cdot p_2,\ p_1 \cdot p_3,\ p_1 \cdot p_4,\ p_2 \cdot p_3,\ p_2 \cdot p_4,\ p_3 \cdot p_4}_{\text{6 invariants, 2 linearly independent, 4 linearly dependent}}.$$

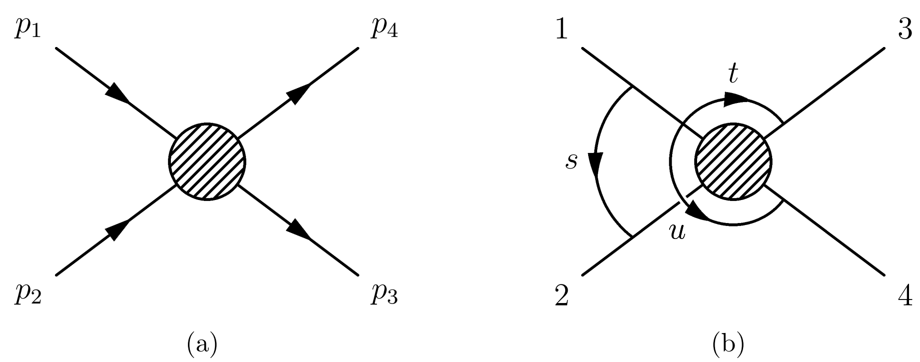

Figure 2.2: *Two-particle scattering.* The kinematical constraints are energy-momentum conservation and the mass shell condition (a). Visualization of Mandelstam variables (b).

Four of them have to be linearly dependent, since there are only two degrees of freedom in the system (center of mass energy and scattering angle).

We now define the Mandelstam variables (see Fig. 2.2(b))

$$\begin{aligned} s &= (p_1 + p_2)^2 \\ t &= (p_1 - p_3)^2 \\ u &= (p_1 - p_4)^2, \end{aligned}$$

where s denotes total center of mass energy squared (positive) and t is the four-momentum transfer squared (negative). Note also that $s + t + u = \sum_{i=1}^{4} m_i^2$.

The center of mass frame is defined by

$$\vec{p}_1 + \vec{p}_2 = 0 = \vec{p}_3 + \vec{p}_4. \tag{2.8}$$

The corresponding variables are asterisked: (cm., $p_i = p_i^*$). The laboratory frame is defined by $\vec{p}_2 = 0$ (fixed target) and variables are labelled with an L : (lab., $p_i = p_i^L$). In deep inelastic scattering the Breit system ($p_i = p_i^B$) is used, which is defined by $\vec{p}_1 + \vec{p}_3 = 0$.

In the following we take a closer look at the center of mass frame, see Fig. 2.3. Equation (2.8) leads to

$$\begin{aligned} \vec{p}_1^{\,*} &= -\vec{p}_2^{\,*} = \vec{p} \\ \vec{p}_3^{\,*} &= -\vec{p}_4^{\,*} = \vec{p}\,' \\ p_1 &= \left(E_1^* = \sqrt{\vec{p}^{\,2} + m_1^2}, \vec{p}\right) \\ p_2 &= \left(E_2^* = \sqrt{\vec{p}^{\,2} + m_2^2}, -\vec{p}\right) \\ p_3 &= (E_3^*, \vec{p}\,') \\ p_4 &= (E_4^*, -\vec{p}\,'). \end{aligned}$$

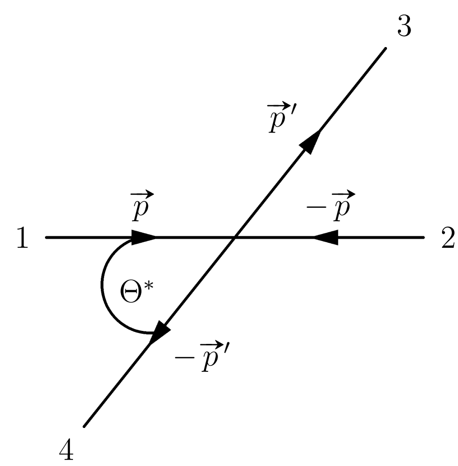

Figure 2.3: *Two-particle scattering in center of mass frame.* For the constraints on the scattering angle Θ^* see section 2.2.1.

The sum

$$p_1 + p_2 = (\underbrace{E_1^* + E_2^*}_{\sqrt{s}}, \vec{0})$$

is no Lorentz invariant, whereas

$$s = (p_1 + p_2)^2 = (E_1^* + E_2^*)^2$$

is one. Now we can express E_i^*, $|\vec{p}|$, and $|\vec{p}\,'|$ in terms of s (see exercise sheet 1):

$$E_{1,3}^* = \frac{1}{2\sqrt{s}}(s + m_{1,3}^2 - m_{2,4}^2) \tag{2.9}$$

$$\vec{p}^{\,2} = (E_1^*)^2 - m_1^2 = \frac{1}{4s}\lambda(s, m_1^2, m_2^2), \tag{2.10}$$

where we have used the Källen function (triangle function) which is defined by

$$\begin{aligned}\lambda(a,b,c) &= a^2 + b^2 + c^2 - 2ab - 2ac - 2bc \\ &= \left[a - (\sqrt{b} + \sqrt{c})^2\right]\left[a - (\sqrt{b} - \sqrt{c})^2\right] \\ &= a^2 - 2a(b+c) + (b-c)^2.\end{aligned}$$

We can see that the Källen function has the following properties:

- symmetric under $a \leftrightarrow b \leftrightarrow c$ and
- asymptotic behavior: $a \gg b, c : \lambda(a, b, c,) \to a^2$.

This enables us to determine some properties of scattering processes. From $\vec{p}^{\,2}, \vec{p}^{\,\prime 2} > 0$ it follows that

$$s_{\text{min}} = \max\left\{(m_1+m_2)^2, (m_3+m_4)^2\right\} \geq 0$$

is the threshold of the process in the s-channel. In the high energy limit ($s \gg m_i^2$) Eq. (2.9) and (2.10) simplify because of the asymptotic behavior of λ and one obtains:

$$E_1^* = E_2^* = E_3^* = E_4^* = |\vec{p}| = |\vec{p}^{\,\prime}| = \frac{\sqrt{s}}{2}.$$

2.2.1 Scattering angle

In the center of mass frame, the scattering angle Θ^* is defined by

$$\vec{p} \cdot \vec{p}^{\,\prime} = |\vec{p}| \cdot |\vec{p}^{\,\prime}| \cos\Theta^*.$$

We also know that

$$\begin{aligned} p_1 \cdot p_3 &= E_1^* E_3^* - |\vec{p}_1^{\,*}||\vec{p}_3^{\,*}| \cos\Theta^* \\ t &= (p_1 - p_3)^2 = m_1^2 + m_3^2 - 2p_1 p_3 = (p_2 - p_4)^2 \end{aligned}$$

and can derive $\cos\Theta^* = \text{function}(s, t, m_i^2)$:

$$\cos\Theta^* = \frac{s(t-u) + (m_1^2 - m_2^2)(m_3^2 - m_4^2)}{\sqrt{\lambda(s, m_1^2, m_2^2)}\sqrt{\lambda(s, m_3^2, m_4^2)}}.$$

This means that $2 \to 2$ scattering is described by two independent variables:

$$\sqrt{s} \text{ and } \Theta^* \qquad\qquad \text{or} \qquad\qquad \sqrt{s} \text{ and } t.$$

2.2.2 Elastic scattering

We now consider the case of elastic scattering. This means that $m_1 = m_3$ and $m_2 = m_4$ (e. g. $ep \to ep$). Therefore Eq. (2.9) and (2.10) simplify:

$$\begin{aligned} &E_1^* = E_3^*, \;\; E_2^* = E_4^* \\ &|\vec{p}|^2 = |\vec{p}^{\,\prime}|^2 = \frac{1}{4s}\Big(s - (m_1+m_2)^2\Big)\Big(s - (m_1-m_2)^2\Big) \end{aligned}$$

and we find for the scattering angle (in the case of elastic scattering)

$$\begin{aligned} t &= (p_1 - p_3)^2 = -(\vec{p}_1 - \vec{p}_3)^2 = -2\vec{p}^{\,2}(1 - \cos\Theta^*) \\ &\Rightarrow \boxed{\cos\Theta^* = 1 + \frac{t}{2|\vec{p}|^2}}. \end{aligned}$$

Restriction to the physically valid region yields

$$\left.\begin{array}{c} -1 \leq \cos\Theta^* \leq 1 \\ \vec{p}^{\,2} \geq 0 \end{array}\right\} \Leftrightarrow \left\{\begin{array}{c} -4|\vec{p}|^2 \leq t \leq 0 \\ s \geq (m_1+m_2)^2 \end{array}\right. .$$

2.2.3 Angular distribution

Finally, we find for the angular distribution (bearing in mind that the distribution is rotationally invariant with respect to the "$\vec{p}$-axis" such that $\int d\phi = 2\pi$)

$$\begin{aligned} d\Omega^* &= 2\pi d\cos\Theta^* \\ \frac{d\Omega^*}{dt} &= \frac{4\pi s}{\sqrt{\lambda(s,m_1^2,m_2^2)}\sqrt{\lambda(s,m_3^2,m_4^2)}} = \frac{\pi}{|\vec{p}||\vec{p}\,'|}. \end{aligned} \tag{2.11}$$

2.2.4 Relative velocity

At this point, we introduce the relative velocity, which we will see to be of relevance in defining the particle flux and hence the collider construction,

$$v_{12} = |\vec{v}_1 - \vec{v}_2| = \left|\frac{\vec{p}_1}{E_1} - \frac{\vec{p}_2}{E_2}\right| = \left|\frac{\vec{p}_1^{\,*}}{E_1^*} - \frac{\vec{p}_2^{\,*}}{E_2^*}\right| = \frac{|\vec{p}_1^{\,*}|}{E_1^* E_2^*}\underbrace{(E_1^* + E_2^*)}_{\sqrt{s}}, \tag{2.12}$$

from which we get,

$$\begin{aligned} v_{12}E_1^*E_2^* &= \sqrt{s}|\vec{p}_1^{\,*}| = \sqrt{s}\sqrt{E_1^{*2} - m_1^2} \\ &= \sqrt{s}\sqrt{\frac{1}{4s}(s+m_1^2-m_2^2) - m_1^2} \\ &= \sqrt{(p_1\cdot p_2)^2 - m_1^2 m_2^2}, \end{aligned} \tag{2.13}$$

the so called Møller flux factor. In going from the first line to the second, we used the definition of the Källen function and in going to the third the fact that $s = m_1^2 + m_2^2 + 2p_1 \cdot p_2$.

We stress here that $v_{12}E_1^*E_2^*$ is a frame independent quantity. It appears in the definition of the incoming particle flux, an thus in the cross section. It also plays an important role in the normalization issues, since the classical volume element is not Lorentz invariant.

2.2.5 Center of mass and laboratory systems

For the center of mass and the laboratory systems respectively, we have,

$$\begin{aligned} CM : s &= (E_1^* + E_2^*)^2 = (\text{total energy})^2 \\ L : s &= m_1^2 + m_2^2 + 2m_2E_1^L \overset{E_1^L \gg m_1, m_2}{\longrightarrow} 2m_2E_1^L. \end{aligned}$$

As an example for the difference, we look at the two operating modes of the Tevatron at Fermilab (Figure 2.4). The energy of the beam particles is $E_{\text{beam}} = 980\,\text{GeV}$.

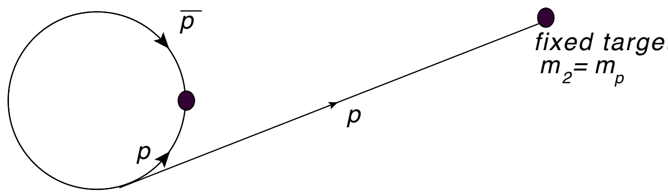

Figure 2.4: *Sketch of the Tevatron accelerator at Fermilab.*

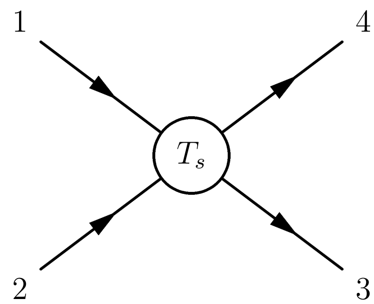

Figure 2.5: *s-channel.*

Used in the $p\bar{p}$-mode, the collision is head on and we are allowed to consider ourselves in the center of mass frame and,

$$\sqrt{s_{p\bar{p}}}(\text{Collider}) = 1960\,\text{GeV},$$

which is ideal for discovering new phenomena with the highest possible energy.

If on the other hand, the pN-mode is chosen (N is a nucleus in the target), we speek of the laboratory frame where

$$\sqrt{s_{pN}}(\text{Fixed target}) = 42.7\,\text{GeV} < m_W.$$

Although this mode is less energetic, it is then possible to create a secondary beam. With this method, the existence of ν_τ could be proven.

2.3 Crossing symmetry

The $2 \to 2$ scattering process has some underlying symmetries, which we shall explore now.

Example When we exchange p_3 and p_4, s is not affected but t and u interchange their roles.

We take now a look at the reaction (Figure 2.5), $1+2 \to 3+4$, for which the 4-momentum is conserved :

$$p_1 + p_2 = p_3 + p_4.$$

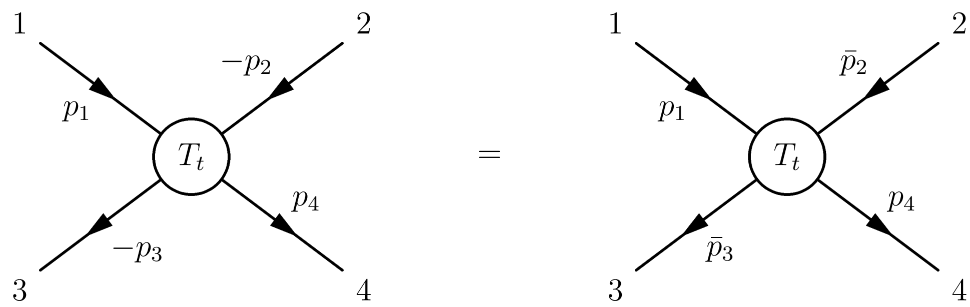

Figure 2.6: *t-channel.*

It is called "s-channel" reaction, because the only positive Mandelstam variable is s. T_s describes the scattering dynamics of the process and will be treated later. It depends on the three Mandelstam variables and is predicted by theoretically (QED, QCD, EW, SUSY,...),

$$T_s(s,t,u) = T(s,t,u)|_{s>0,t\leq 0,u\leq 0}. \tag{2.14}$$

T can then be extended analytically to the whole range $s,t,u \in \mathbb{R}$. Depending on the region, it can then describe different **crossed reactions**.

For instance, suppose we exchange p_2 and p_3, we then get naively (Figure 2.6),

$$p_1 + (-p_3) = (-p_2) + p_4.$$

We now make the interpretation

$$-p_n = p_{\bar{n}},$$

in which $\bar{n}$ stands for the antiparticle of the particle n, leading to the expression (Figure 2.6),

$$p_1 + p_{\bar{3}} = p_{\bar{2}} + p_4.$$

Since 1 and $\bar{3}$ are the incoming particles, we speak of the "t-channel" process. One has

$$T_t(s,t,u) = T(s,t,u)|_{s\leq 0,t>0,u\leq 0}. \tag{2.15}$$

2.3.1 Interpretation of antiparticle-states

As stated above, we interpret particles with 4-momentum $-p$ to be antiparticles with 4-momentum p. The reason for that becomes clear when we look at the 4-current,

$$j^\mu \overset{ED}{=} \begin{pmatrix} \rho \\ \vec{j} \end{pmatrix} \overset{QM}{=} \underbrace{\underbrace{-e}_{\text{electron charge}} \quad \underbrace{i(\varphi^*\partial^\mu\varphi - \varphi\partial^\mu\varphi^*)}_{\text{probability density}}}_{\text{charge density}}. \tag{2.16}$$

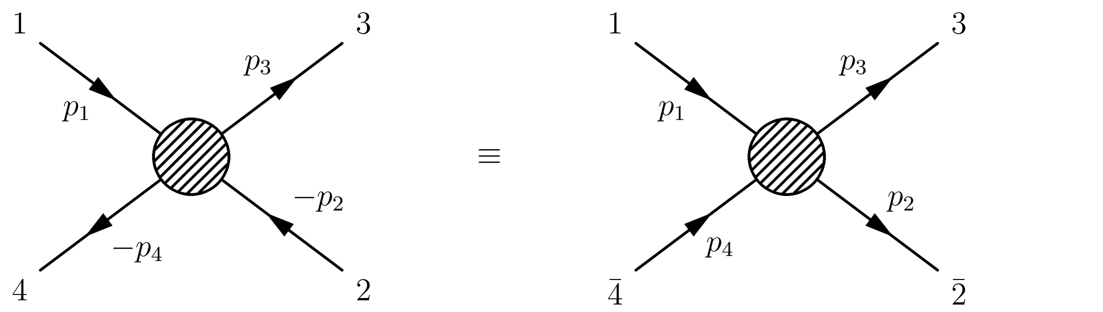

Figure 2.7: *Emission of a positron and absorption of an electron.* The emission of a positron with energy $+E$ is equivalent to the absorption of an electron with energy $-E$.

Inserting the wave function of the free electron,

$$\varphi = N\mathrm{e}^{-ip\cdot x}, \tag{2.17}$$

in the definition of the 4-current Eq. (2.16), one gets

$$\begin{aligned}
e^- \text{ with 4-momentum} + p^\mu &: j^\mu(e^-) = -2e|N|^2 p^\mu = -2e|N|^2 \begin{pmatrix} +E \\ +\vec{p} \end{pmatrix}, \\
e^+ \text{ with 4-momentum} + p^\mu &: j^\mu(e^+) = +2e|N|^2 p^\mu = -2e|N|^2 (-p)^\mu, \\
e^- \text{ with 4-momentum} - p^\mu &: j^\mu(e^-) = -2e|N|^2 (-p)^\mu = -2e|N|^2 \begin{pmatrix} -E \\ -\vec{p} \end{pmatrix},
\end{aligned}$$

and hence the rule,

$$\boxed{j^\mu(e^+) = j^\mu(e^-) \text{ with the subsititution } p^\mu \to -p^\mu}. \tag{2.18}$$

We stress here the fact that the *whole* 4-vector p^μ takes a minus sign, and not only the spatial part $\vec{p}$.

What we effectively used here is the fact that in the phase of Eq. (2.17) we can flip the signs of both p^μ and x^μ without changing the wave function. There is no place here for particle travelling backwards in time!

A particle with 4-momentum $-p^\mu$ is a *representation* for the corresponding *anti*particle with 4-momentum p^μ. Alternatively, one can say that the emission of a positron with energy $+E$ corresponds to the absorption of an electron with energy $-E$. Figure 2.7 restates the last sentence as a Feynman diagram.

The three reactions (s-, t- and u-channels) are described by a single function $T(s, t, u)$ evaluated in the relevant kinematical region ($s \geq 0$ or $t \geq 0$ or $u \geq 0$).

In order to represent the situation, one usually refers to the **Dalitz plot**[1] (Figure 2.8).

[1] or equilateral coordinates

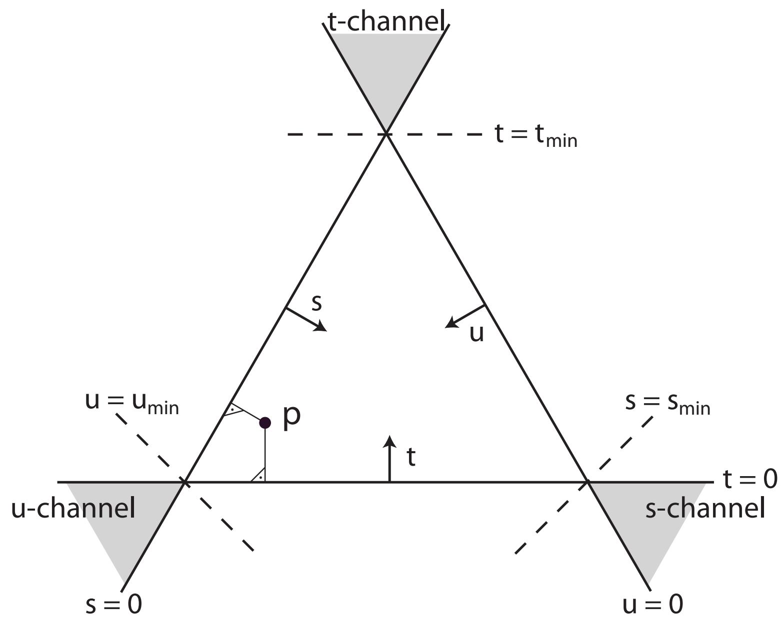

Figure 2.8: *Dalitz plot of s-, t-, and u-channels.*

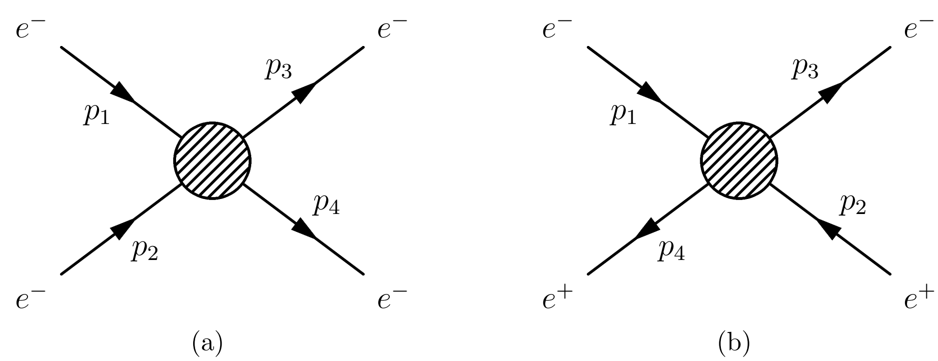

Figure 2.9: *Møller scattering (a) and Bhabha scattering (b).*

Example We take a look at the **Møller scattering**,

$$e^-e^- \to e^-e^-,$$

which is the s-channel of the reaction depicted on Figure 2.9(a). By crossing, we get as u-channel reaction the **Bhabha scattering**,

$$e^+e^- \to e^+e^-,$$

which is the reaction depicted on Figure 2.9(b).

The considerations of this chapter enable us to derive *constraints* on the possible dynamics but are not sufficient to *decide* on the dynamics. To "get" the dynamics we must calculate and compare to experiments decay rates and scattering cross-sections.