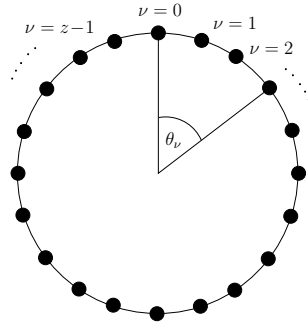


Exercise 2.1 Statistical Polarization

We study a system of N compasses with a needle pointing in one of z directions. The needle hops to a neighboring direction with a rate Γ (nearest neighbor interaction). We assume that z is even for ease of presentation.



- a) Express the master equation and the entropy (H -function) of this system in terms of the numbers N_ν of compasses pointing in direction ν , where $\nu \in \{0, 1, \dots, z-1\}$. What is the equilibrium state of this system?
- b) The system is prepared with all the needles pointing uniformly distributed to the positions $\nu \in \{0, 1, \dots, z/2-1\}$. What is the entropy of the initial distribution? Compare your result with the equilibrium entropy and interpret the result with respect to the ideal gas entropy.
- c) At time $t = 0$, all needles point in the same direction ($N_0 = N, N_\nu = 0 \forall \nu \neq 0$). Calculate for long times ($t \gg z^2/\Gamma$) the polarization of the system

$$P(t) := \langle \cos \theta \rangle(t) = \frac{1}{N} \sum_{\nu} N_{\nu}(t) \cos \theta_{\nu}, \quad \theta_{\nu} = \frac{2\pi\nu}{z}, \quad (1)$$

and compare the relaxation of the polarization with the one for the entropy.

Hint: Use the Fourier transform

$$N_{\nu}(t) = \sum_{k=-z/2}^{z/2-1} \tilde{N}_k(t) e^{-i\frac{2\pi}{z}k\nu}$$

and express the master equation and $P(t)$ in terms of the coefficients $\tilde{N}_k(t)$.

- d) Starting with the same initial distribution as in c), calculate the exact time dependence of N_ν for the case of $z \rightarrow \infty$.

Hint: The N_ν can be written in terms of Bessel-functions using the Jacobi-Anger expansion

$$e^{ix \cos \theta} = \sum_{n=-\infty}^{\infty} i^n J_n(x) e^{in\theta}. \quad (2)$$

- e) We go to a continuum description: $z \rightarrow \infty$ with $L = za$ constant, where a is the distance between points. We find the diffusion equation

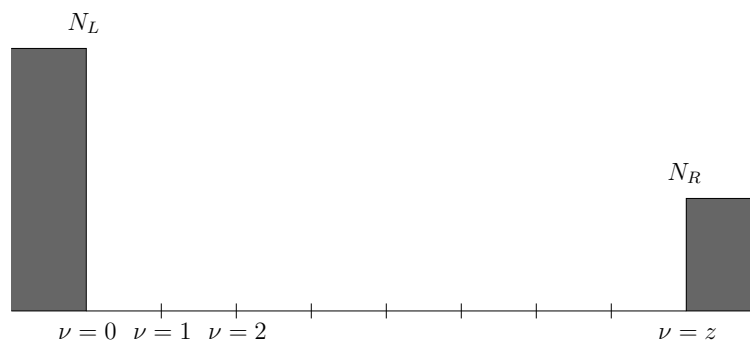
$$\dot{\rho}(x, t) = D\partial_x^2\rho(x, t). \quad (3)$$

What is D in this equation?

Starting with the same initial distribution as in c), calculate the time dependence of the entropy for $L \rightarrow \infty$ (i.e. ignoring boundary effects). Interpret your result with respect to the ideal gas.

Exercise 2.2 Particle Current and Entropy Production

We consider a chain with z sites that is connected to two reservoirs such that the boundary conditions $N_0 = N_L$ and $N_z = N_R$ hold for all times (see figure).



- a) Find the stationary solution and show that the entropy is not a constant in this case. What process is responsible for the increase in entropy?
- b) We can define a particle current J_ν on the bond between the sites ν and $\nu + 1$ through the continuity equation $\dot{N}_\nu = J_{\nu+1} - J_\nu$. Consider the stationary state close to the equilibrium, where

$$N_L - N_R = \epsilon, \quad 0 < \epsilon \ll N_L.$$

Show that while the current is proportional to ϵ , the entropy production is proportional to ϵ^2 . What does that mean for the description of transport phenomenon (heat, particles) close to equilibrium?

Office Hours: Monday, 09/28, 9 – 10 at HIT K 33.3.