Quantenmechanik I
Lösungsblatt 8

HS 2009
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Exercise 8.1 Tensor products

a) Suppose $\lambda = \sum_{k=1}^{M} d_k(\eta_k \otimes \mu_k)$. Then $\{\eta_k\}_{k=1}^{M}$ and $\{\mu_k\}_{k=1}^{M}$ span subspaces $M_1 \subset \mathcal{H}_1$ and $M_2 \subset \mathcal{H}_2$ respectively. If we let $\{\phi_j\}_{j=1}^{N_1}$ and $\{\psi_l\}_{l=1}^{N_2}$ be orthonormal bases for M_1 and M_2, we can express each η_k in terms of the ϕ_j's and each μ_k in terms of the ψ_l's obtaining

$$\lambda = \sum_{j=1,l=1}^{M_1,M_2} c_{jl}(\phi_j \otimes \psi_l) \tag{1}$$

But,

$$\begin{aligned}(\lambda, \lambda) &= \left(\sum c_{jl}(\phi_j \otimes \psi_l), \sum c_{im}(\phi_i \otimes \psi_m)\right) &(2)\\ &= \sum \overline{c_{jl}} c_{im}(\phi_j, \phi_i)(\psi_l, \psi_m) &(3)\\ &= \sum_{jl} |c_{jl}|^2 &(4)\end{aligned}$$

so if $(\lambda, \lambda) = 0$, then all the $c_{jl} = 0$ and λ is the zero form. Thus $(\cdot, \cdot)$ is positive definite.

b) Let $\{\xi_r\}$ be an orthonormal basis of an Hilbert space $\mathcal{H}$, and $\sum_r |c_r|^2 < \infty, c_r \in \mathbb{C}$, then $\sum_r c_r \xi_r$ converges to an element of $\mathcal{H}$. The set $\{\phi_k \otimes \psi_l\}$ is clearly orthonormal and therefore we need only to show that $\mathcal{E} \subseteq S = \overline{span\{\phi_k \otimes \psi_l\}}$, because $\mathcal{E}$ is by construction dense in $\mathcal{H}_1 \otimes \mathcal{H}_2$. Thus if $\mathcal{E}$ is contained in the closed set spanned by the $\phi_k \otimes \psi_l$, then $\overline{span\{\phi_k \otimes \psi_l\}} = \mathcal{H}_1 \otimes \mathcal{H}_2$. Let $\phi \otimes \psi \in \mathcal{E}$—it is enough to consider elements of the form $\phi \otimes \psi$ since all other elements in $\mathcal{E}$ can be written as finite linear combinations of these elements. Then $\phi = \sum a_k \phi_k$ and $\psi = \sum b_l \psi_l$ with $\sum_k |a_k|^2 < \infty$ and $\sum_l |b_l|^2 < \infty$. Hence $\sum_{k,l} |a_k b_l|^2 < \infty$. Therefore $\mu = \sum_{k,l} a_k b_l \phi_k \otimes \psi_l \in S$ and $\|\phi \otimes \psi - \sum_{k<K,l<L} a_k b_l \phi_k \otimes \psi_l\| \mapsto 0$ as $K, L \mapsto \infty$.

c) Suppose that $f(x, y) \in L^2(M_1 \times M_2, d\mu_1 d\mu_2)$, and

$$\int\int_{M_1 \times M_2} \overline{f(x, y)} \phi_k(x) \psi_l(y) d\mu_1(x) d\mu_2(y) = 0 \tag{5}$$

for all k and l. By Fubini's theorem this can be rewritten

$$\int_{M_2} \left(\int_{M_1} \overline{f(x, y)} \phi_k(x) d\mu_1(x)\right) \psi_l(y) d\mu_2(y) \tag{6}$$

$$= \left(\overline{\int_{M_1} \overline{f(x, y)} \phi_k(x) d\mu_1(x)}, \psi_l\right)_{L^2} = 0 \tag{7}$$

Since $\{\psi_l\}$ is a basis for $L^2(M_2, d\mu_2)$, this implies that

$$\int_{M_1} \overline{f(x, y)} \phi_k(x) d\mu_1(x) = (f(x, y), \phi_k)_{L^2} = 0 \tag{8}$$

exept on a set $S_k \subset M_2$ with $\mu_2(S_k) = 0$. Thus, for $y \notin \bigcup S_k$, $\int_{M_1} f(x, y) \phi_k(x) d\mu_1(x) = 0$ for all k, which implies that $f(x, y) = 0$ a.e. $[\mu_1]$. Thus, $f(x, y) = 0$ a.e. $[\mu_1 \mu_2]$. So, $\{\phi_k(x)\psi_l(y)\}$ is a basis for $L^2(M_1 \times M_2, d\mu_1 d\mu_2)$

d) Define

$$U : \phi_k \otimes \psi_l \mapsto \phi_k(x)\psi_l(y) \tag{9}$$

Then U takes an orthonormal basis for $L^2(M_1, d\mu_1) \otimes L^2(M_2, d\mu_2)$ onto an orthonormal basis for $L^2(M_1 \times M_2, d\, u_1 d\mu_2)$ and extends uniquely, via the $B.L.T$ theorem[1], to a unitary mapping of $L^2(M_1, d\mu_1) \otimes L^2(M_2, d\mu_2)$ onto $L^2(M_1 \times M_2, d\mu_1 d\mu_2)$.

e) Let $\{\phi_k\}$ and $\{\psi_l\}$ be orthonormal bases for $\mathcal{H}_1$ and $\mathcal{H}_2$ and suppose $\sum c_{kl}\phi_k \otimes \psi_l$ is a finite sum. Then

$$\begin{aligned} \|(A \otimes \mathbb{1}) \sum c_{kl}(\phi_k \otimes \psi_l)\|^2 &= \sum_l \| \sum_k c_{kl} A\phi_k\|^2 && (10)\\ &\leq \sum_l \|A\|^2 \sum_k |c_{kl}|^2 = \|A\|^2 \| \sum c_{kl}\phi_k \otimes \psi_l\|^2 && (11) \end{aligned}$$

Since the set of such finite sums is dense in $\mathcal{H}_1 \otimes \mathcal{H}_2$, we conclude that $\|A \otimes \mathbb{1}\| \leq \|A\|$. Thus, with $A \otimes B = (A \otimes \mathbb{1})(\mathbb{1} \otimes B)$, we have $\|A \otimes B\| \leq \|A \otimes \mathbb{1}\|\|\mathbb{1} \otimes B\| \leq \|A\|\|B\|$, with the first inequality following form the definition of the operator norm.

Conversely, given $\epsilon > 0$, there exists unit vectors $\phi \in \mathcal{H}_1, \psi \in \mathcal{H}_2$, so that $\|A\phi\| \geq \|A\| - \epsilon$ and $\|B\psi\| \geq \|B\| - \epsilon$. Then,

$$\|(A \otimes B)(\phi \otimes \psi)\| = \|A\phi\|\|B\psi\| \geq \|A\|\|B\| - \epsilon\|A\| - \epsilon\|B\| + \epsilon^2. \tag{12}$$

Since $\epsilon > 0$ is arbitrary $\|A \otimes B\| \geq \|A\|\|B\|$ which concludes the proof.

Exercise 8.2 Spectra of non-commuting unbounded observables

a) Define

$$f(x) = exp(xA)Bexp(-xA) \tag{13}$$

We expand $f(x)$ as a Taylor series in x about the origin. From the definition

$$f'(x) = exp(xA)(AB - BA)exp(-xA) \tag{14}$$

so that $f'(0) = [A, B]$.

$$\begin{aligned} f''(x) &= exp(xA)(A[A,B] = [A,B]A)exp(-xA), && (15)\\ f''(0) &= [A,[A,B]], && (16) \end{aligned}$$

etc. We now write the Taylor series for $f(x)$,

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + ... \tag{17}$$

and substitute for $f^{(n)}(0)$, which immediately leads to the series

$$exp(xA)Bexp(-xA) = B + x[A,B] + \frac{x^2}{2!}[A,[A,B]] + \dots \,. \tag{18}$$

The statement follows when we replace x with ix. Remark: Though this calculation appears to be very simple, there is a subtlety we ommited. In principle one has to make sure that the domains of A and $[A, B]$, A and $[A, [A.B]]$, etc. are compatible.

[1] The B.L.T (bounded linear transformation) theorem allows to uniquely extend a bounded linear transformation defined on a dense subset of the Hilbert space to the entire Hilbert space.

b) With $[A, B] = -i\mathbb{1}$ and part a) it follows that $exp(ixA)Bexp(-ixA) = B + x\mathbb{1}$, i.e., $exp(ixA)$ can be interpreted as a displacement operator. Think also of the analogy with Q and P. We use this to show that the assumption of a point spectrum leads to a contradiction: Let ψ be an eigenvector of B corresponding to the eigenvalue β. Then $\phi := exp(-ixA)\psi$ is eigenvector of B corresponding to the eigenvalue $\beta + x$:

$$\begin{aligned} B\phi &= exp(-ixA)exp(ixA)Bexp(-ixA)\psi && (19)\\ &= exp(-ixA)(B + x\mathbb{1})\psi = exp(-ixA)(\beta + x)\psi && (20)\\ &= (\beta + x)exp(-ixA)\psi = (\beta + x)\phi && (21)\end{aligned}$$

Since $x \in \mathbb{R}$ is arbitrary, and eigenvectors of different eigenvalues are linear independent, we have found uncountable many linear independent eigenvectors, which is impossible in a separable Hilbert space. Hence the spectrum must be continuous.

c) We want to show that, starting from any point E in the continuous spectrum, we can shift E by any real number and still reach an element of the spectrum. So, let $E \in \sigma(A)$, then we can write formally, without specifying the series $\psi_n \in \mathcal{H}$,

$$\|(A + a - E)\psi\| = \|exp(iaA)(A - E)exp(-iaA)\psi\| = \|(A - E)exp(-iaA)\psi\| \qquad (22)$$

The last equality follows because $exp(iaA)$ is unitary and the norm of vector does not change under unitary transformations.

The *Weyl-criterion* states: *"E is part of the continuous spectrum of $\mathcal{H}_r$ ⇔ It exists a Weyl-sequence $\{\psi_n\}$ corresponding to $(\mathcal{H}_r, E)$"*. Hence we can find a sequence ψ'_n with $\|(H - E)\psi'_n\| \overset{n\to\infty}{\longrightarrow} 0$. Thus $\|(A + a - E)\psi_n\| = \|(A - E)exp(-iaA)\psi_n\| \overset{n\to\infty}{\longrightarrow} 0$ for $\psi_n = exp(iaA)\psi'_n$ and immediately, again by the Weyl-criterion, $E - a \in \sigma(A)$ because of (22). The conjecture is proved, since $a \in \mathbb{R}$ was arbitrary.

An alternative solution is to use the unitary invariance of the spectrum, i.e., we know that $\sigma(B) = \sigma(exp(ixA)Bexp(-ixA)) = \sigma(B + x\mathbb{1})$. However, the spectrum of $B + x$ is just the spectrum of B shifted by an arbitrary $x \in \mathbb{R}$, so the spectrum of B is invariant under aribtrary shifts. Hence $\sigma(B) = \mathbb{R}$.

Exercise 8.3 Spin systems

a) linear algebra $\surd$

b) We set $\sigma_4 := \mathbb{1}$. The space of all 2×2 hermitian matrices is a reel vector space V. We introduce the Hilbert-Schmidt inner product $\langle A, B\rangle = Tr(A^*B)$ on V. The Pauli-matrices together with $\mathbb{1}$ form a basis, so we can take any hermitian matrix, say a density matrix, and expand it in the basis. From part a) we know that the Pauli-matrices are not normalized, so we have to add a factor $\frac{1}{\sqrt{2}}$,

$$\rho = \sum_{i=1}^{4} \langle \rho, \frac{1}{\sqrt{2}}\sigma_i\rangle \frac{1}{\sqrt{2}}\sigma_i = \sum_{i=1}^{4} \left(\frac{1}{\sqrt{2}}a_i\right) \frac{1}{\sqrt{2}}\sigma_i, \qquad (23)$$

with $a_i := \langle \rho, \sigma_i\rangle$. For a pure state we know that $Tr(\rho^2) = Tr(\rho) = 1$ and thus, by calculating $\langle \rho, \rho\rangle$ from (23), $\sum_{i=1}^{4} a_i^2 = 2$. In particular we have

$$\sum_{i=1}^{3} a_i^2 = 1, \qquad (24)$$

because $\langle \rho, \mathbb{1}\rangle = Tr(\rho\mathbb{1}) = 1$.

We rewrite the expression in b) a bit

$$\langle\psi,\vec{\sigma}\psi\rangle = \sum_{i=1}^{3}\langle\psi,\vec{\sigma}_i\psi\rangle_{\mathcal{H}}\hat{e}_i = \sum_{i=1}^{3} Tr\left(|\psi\rangle\langle\psi|\sigma_i\right)\hat{e}_i \tag{25}$$

$$= \sum_{i=1}^{3}\langle\rho,\sigma_i\rangle_{\mathcal{H}_{HS}}\hat{e}_i = \sum_{i=1}^{3} a_i\hat{e}_i \tag{26}$$

The statement then follows from (24). □

c) For pure states, the statement essentially follows b): We can rewrite (23) a bit

$$\sum_{i=1}^{4}\left(\frac{1}{\sqrt{2}}a_i\right)\frac{1}{\sqrt{2}}\sigma_i = \frac{1}{2}\mathbb{1} + \sum_{i=1}^{3}\left(\frac{1}{\sqrt{2}}a_i\right)\frac{1}{\sqrt{2}}\sigma_i \tag{27}$$

$$= \frac{1}{2}\mathbb{1} + \frac{1}{2}\sum_{i=1}^{3} a_i\sigma_i \tag{28}$$

with $\vec{a}$ being a unit vector. For mixed states $Tr(\rho^2) < 1$, and the argument right before (24) tells us that $\sum_{i=1}^{4} a_i^2 < 2$ and thus

$$\sum_{i=1}^{3} a_i^2 < 1. \tag{29}$$

□

Remark: This little calculation shows that we can parametrize density matrices of two-level systems by a unit sphere, the so-called **Bloch-sphere**. Pure states lie on the surface of the sphere, while mixed states fill the inside of the sphere. The totally mixed state lies in the center. The concept of the Bloch-sphere can be handy to visualize effects of quantum operations on two-level systems.

d) The claim is actually just a rephrasing of c): It follows from (24) and (29) together with

$$|Tr(\vec{\sigma}\rho)| = \left|\begin{pmatrix}\langle\rho,\sigma_1\rangle\\ \langle\rho,\sigma_2\rangle\\ \langle\rho,\sigma_3\rangle\end{pmatrix}\right| = \left|\begin{pmatrix}a_1\\ a_2\\ a_3\end{pmatrix}\right| = \sqrt{\sum_{i=1}^{3} a_i^2} \tag{30}$$

e)

$$\rho = |\psi^-\rangle\langle\psi^-| = \frac{1}{2}\begin{pmatrix}0 & 0 & 0 & 0\\ 0 & +1 & -1 & 0\\ 0 & -1 & +1 & 0\\ 0 & 0 & 0 & 0\end{pmatrix} \tag{31}$$

f) We use $\langle\psi^-,\sigma_z\otimes(\vec{\sigma}\vec{a})\psi^-\rangle = Tr\left(|\psi^-\rangle\langle\psi^-|,\sigma_z\otimes\left(\sum_i a_i\sigma_i\right)\right)$ to calculate the expecation value. First

$$a_1\sigma_x + a_2\sigma_y + a_3\sigma_z = \begin{pmatrix}0 & a_1\\ a_1 & 0\end{pmatrix} + \begin{pmatrix}0 & -ia_2\\ ia_2 & 0\end{pmatrix} + \begin{pmatrix}a_3 & 0\\ 0 & -a_3\end{pmatrix} = \begin{pmatrix}a_3 & a_1 - ia_2\\ a_1 + ia_2 & -a_3\end{pmatrix} \tag{32}$$

and

$$\sigma_z\otimes(\vec{\sigma}\vec{a}) = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\otimes\begin{pmatrix}a_3 & a_1 - ia_2\\ a_1 + ia_2 & -a_3\end{pmatrix} = \begin{pmatrix}a_3 & a_1 - ia_2 & 0 & 0\\ a_1 + ia_2 & -a_3 & 0 & 0\\ 0 & 0 & -a_3 & -a_1 + ia_2\\ 0 & 0 & -a_1 - ia_2 & a_3\end{pmatrix} \tag{33}$$

and

$$Tr\left(\frac{1}{2}\begin{pmatrix}0 & 0 & 0 & 0\\ 0 & +1 & -1 & 0\\ 0 & -1 & +1 & 0\\ 0 & 0 & 0 & 0\end{pmatrix}\begin{pmatrix}a_3 & a_1 - ia_2 & 0 & 0\\ a_1 + ia_2 & -a_3 & 0 & 0\\ 0 & 0 & -a_3 & -a_1 + ia_2\\ 0 & 0 & -a_1 - ia_2 & a_3\end{pmatrix}\right) = -a_3 = -\cos(\theta). \quad (34)$$

Exercise 8.4 Riesz's lemma

Boundedness and linearity imply continuity. The preimage of a closed set under a continuous function is closed, hence $\ker(\ell)$ is closed since $\{0\}$ is closed. For Hilbert-spaces this immediately implies the existence of a non-zero vector in the orthogonal complement. The proof of this statement relies on the completeness of the Hilbert space and the existence of a scalar-product. It is enough to consider $\ker(\ell) \subset \mathcal{H}$ for if $\ker(\ell)$ was the full Hilbert space then ℓ would be zero on all vectors and thus $\vec{0}$ does exactly what we want. Hence assume $\ker(\ell) \subset \mathcal{H}$ and define $\psi_\ell = \overline{\ell(\phi_0)}\|\phi_0\|^{-2}\phi_0$, $\phi_0 \in \ker(\ell)^\perp$. Let ϕ be in $\ker(\ell)$. Then

$$\ell(\phi) = 0 = \langle \psi_\ell, \phi \rangle. \quad (35)$$

Now let $\phi = \alpha\phi_0$, then

$$\ell(\phi) = \ell(\alpha\phi_0) = \alpha\ell(\phi_0) = \langle \overline{\ell(\phi_0)}\|\phi_0\|^{-2}\phi_0, \alpha\phi_0 \rangle = \langle \psi_\ell, \alpha\phi_0 \rangle \quad (36)$$

$\ell(\cdot)$ and $\langle \psi_\ell, \cdot \rangle$ are linear and agree on $\operatorname{span}\{\ker(\ell), \phi_0\}$.
To prove uniquness we assume that there exist two linear independent vectors in $\ker(\ell)^\perp$, ψ_1, ψ_2, with $\ell(\psi_1), \ell(\psi_2) \neq 0$. Define

$$\psi = \psi_1 - \frac{\ell(\psi_1)}{\ell(\psi_2)}\psi_2 \quad (37)$$

Then, as can readily be checked, $\ell(\psi) = 0$, which contradicts linear independence. Therefore $\dim(\ker(\ell)^\perp) = 1$.
To prove that $\|\ell\|_{\mathcal{H}^*} = \|\psi_\ell\|_{\mathcal{H}}$ we observe that

$$\|\ell\| = \sup_{\|\phi\|\leq 1} |\ell(\phi)| = \sup_{\|\phi\|\leq 1} |(\psi_\ell, \phi)| \leq \sup_{\|\phi\|\leq 1} \|\psi_\ell\|\|\phi\| = \|\psi_\ell\|. \quad (38)$$

and

$$\|\ell\| = \sup_{\|\phi\|\leq 1} |\ell(\phi)| \geq \left|\ell\left(\frac{\psi_\ell}{\|\psi_\ell\|}\right)\right| = \left(\psi_\ell, \frac{\psi_\ell}{\|\psi_\ell\|}\right) = \|\psi_\ell\|. \quad (39)$$

□