Quantenmechanik I
Serie 8

Exercise 8.1 Tensor products

In this exercise we want recapitulate the concept of tensor products of Hilbert spaces $\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \dots$ and tensor products of operators $A \otimes B \otimes \dots$. We want to understand how to define operators on tensor products of Hilbert spaces and why the tensor product is a reasonable mathematical framework to describe composite quantum systems.
Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces. For each $\phi_1 \in \mathcal{H}_1$, $\phi_2 \in \mathcal{H}_2$ we define the "symbol" $\phi_1 \otimes \phi_2$ as the conjugate bilinear form which acts on the cross-product $\mathcal{H}_1 \times \mathcal{H}_2$ like

$$(\phi_1 \otimes \phi_2) : \mathcal{H}_1 \times \mathcal{H}_2 \mapsto \mathbb{C}; \quad (\phi_1 \otimes \phi_2)\langle \psi_1, \psi_2 \rangle = (\psi_1, \phi_1)_{\mathcal{H}_1} (\psi_2, \phi_2)_{\mathcal{H}_2}. \tag{1}$$

Let $\mathcal{E}$ denote the set of all finite linear combinations of the forms $\phi_1 \otimes \phi_2$. The inner products $(\cdot,\cdot)_{\mathcal{H}_i}$ on $\mathcal{H}_i$ naturally induce an inner product $(\cdot,\cdot)_{\mathcal{E}}$ on $\mathcal{E}$ through

$$(\phi \otimes \psi, \eta \otimes \mu)_{\mathcal{E}} := (\phi, \eta)_{\mathcal{H}_1} (\psi, \mu)_{\mathcal{H}_2}, \tag{2}$$

which also makes $\mathcal{E}$ a metric space. The tensor product $\mathcal{H}_1 \otimes \mathcal{H}_2$ is defined as the completion of $\mathcal{E}$[1] and thus is itself a Hilbert space.

a) Show that $(\cdot,\cdot)_{\mathcal{E}}$ is positive-definite.

b) Let $\{\phi_k\}$, $\{\psi_l\}$ be bases for $\mathcal{H}_1$, $\mathcal{H}_2$. Show that $\{\phi_k \otimes \psi_l\}$ is a basis for $\mathcal{H}_1 \otimes \mathcal{H}_2$.

c) Consider two separable Hilbert spaces $L^2(M_1, d\mu_1)$, $L^2(M_2, d\mu_2)$. Let $\{\phi_k(x)\}$ and $\{\psi_l(y)\}$ be the respective bases. Show that $\{\phi_k(x)\psi_l(y)\}$ is a basis of $L^2(M_1 \times M_2, d\mu_1 d\mu_2)$.

d) Prove that $L^2(M_1, d\mu_1) \otimes L^2(M_2, d\mu_2) \cong L^2(M_1 \times M_2, d\mu_1 d\mu_2)$ by giving the explicit unitary mapping between the two Hilbert spaces. This motivates the tensor product of Hilbert spaces as a suitable structure for composite quantum systems.

Consider two operators A and B densely defined on $D(A) \subseteq \mathcal{H}_1$ and $D(B) \subseteq \mathcal{H}_2$ respectively. The tensor product $A \otimes B$ defined on the dense set $D(A) \otimes D(B) \subseteq \mathcal{H}_1 \otimes \mathcal{H}_2$ is defined by

$$(A \otimes B)(\sum_i c_i \phi_i \otimes \psi_i) := \sum_i c_i (A\phi_i \otimes B\psi_i). \tag{3}$$

e) Prove that for bounded operators A and B on Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$

$$\|A \otimes B\|_{\mathcal{H}_1 \otimes \mathcal{H}_2} = \|A\|_{\mathcal{H}_1} \|B\|_{\mathcal{H}_2}. \tag{4}$$

Exercise 8.2 Spectra of non-commuting unbounded observables

Consider two unbounded self-adjoint operators A, B with $[A, B] = -i\mathbb{1}$[2]. We want to show that the spectra of A and B necessarily cover the entire real line, but first we have to prove a useful operator extension theorem.

a) Show that $e^{ixA} B e^{-ixA} = B + ix[A, B] - \frac{x^2}{2!}[A, [A, B]] + \dots$.

b) Show that B does not have a point spectrum.

c) Use the *Weyl-Criterion* (lecture notes Appendix C, Satz 5) to show that $\sigma(A) = \sigma(B) = \mathbb{R}$.

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[1] All metric spaces can be completed. See e.g. Ph. Blanchard and E.Brüning, "Distributionen und Hilbertraumoperatoren", Springer, 1993.

[2] We tacitly assume that the domains of A and B are compatible such that the commutator is well-defined.

Exercise 8.3 Spin systems

Consider the Hilbert space $\mathbb{C}^2$ with its usual scalar product $\langle \cdot, \cdot \rangle$. The Pauli-matrices

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{5}$$

together with the identity, form a basis of all 2×2 hermitian matrices. Prove the following statements:

a) $[\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k$; $\quad [\sigma_i, \sigma_j]_+ = 2\delta_{ij}\mathbb{1}$; $\quad Tr(\sigma_i\sigma_j) = 2\delta_{ij}$.

b) For $\psi \in \mathbb{C}^2$ with $\langle \psi, \psi \rangle = 1$, the expectation value $\langle \psi, \vec{\sigma}\psi \rangle = \vec{n} \in S^2$ is a unit vector.

c) Every density matrix ρ can be represented as

$$\rho = \frac{1}{2}(1 + \vec{a}\vec{\sigma}) \text{ with } |\vec{a}| \leq 1. \tag{6}$$

For $|\vec{a}| = 1$, ρ is a projection.

d) For mixed states $|Tr(\vec{\sigma}\rho)| < 1$, for pure states, however, $|Tr(\vec{\sigma}\rho)| = 1$.

Let $\{|\uparrow\rangle, |\downarrow\rangle\}$ denote the canonical z-basis, i.e., the basis in which we wrote the Pauli-matrices in Equation (5). Consider the singlet state $|\psi^-\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$ on the Hilbert space $\mathbb{C}^2 \otimes \mathbb{C}^2$.

e) Give the explicit form of $|\psi^-\rangle$ when written as a 4×4 density matrix.

f) Let θ, with $\vec{a}\vec{e}_z = \cos\theta$, $|\vec{a}| = 1$, denote the angle between two spins. Calculate the expectation value $\langle \psi^-, \sigma_z \otimes (\vec{\sigma}\vec{a})\psi^- \rangle$.

Exercise 8.4 Riesz's lemma

This famous lemma characterizes the *dual space* $\mathcal{H}^*$, the space of all bounded linear forms on a Hilbert space. In a nutshell, the lemma states that every Hilbert space is its own dual space. Prove Riesz's lemma, i.e., prove that every $\ell \in \mathcal{H}^*$, $l : \mathcal{H} \mapsto \mathbb{C}$, can be written as $\ell(\phi) = \langle \psi_\ell, \phi \rangle_\mathcal{H}$, with unique $\psi_\ell \in \mathcal{H}$. Show also that $\|\psi_\ell\|_\mathcal{H} = \|\ell\|_{\mathcal{H}^*}$.
Hints: We explicitly construct the vector $\psi_\ell \in \mathcal{H}$ that fulfills the requirement and then prove its uniqueness. Consider $\ker(\ell) \subseteq \mathcal{H}$. Argue why we can assume that $\ker(\ell)$ is a proper subset of $\mathcal{H}$, i.e., $\ker(\ell) \subset \mathcal{H}$.

i) Argue why $\ker(\ell)$ is a closed subspace[3]*; use that boundedness and linearity imply continuity.*

ii) Existence: Show that $\psi_\ell = \overline{\ell(\phi_0)}\|\phi_0\|^{-2}\phi_0$, $\phi_0 \in \ker(\ell)^\perp$, has the right properties on span$\{\ker(\ell), \phi_0\}$.

iii) Uniqueness: Show that $\ker(\ell)^\perp$ is one-dimensional: take a linear combination of two linear-independent vectors in $\ker(\ell)^\perp$ and lead this to a contradiction.

Remark: A corollary of Riesz's lemma is the *bra-ket* notation.

[3]It is a special property of Hilbert spaces that the closedness of a subspace implies the existence of non-zero vectors in the orthogonal complement of this subspace.