Solutions 6 - Gravitational two body problems

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1. Planets falling into each other

After the reduction to the one body equivalent problem the Lagrangian is

$$L = \frac{\mu}{2}(\dot{r}^2 + r^2\dot{\theta}^2) + \frac{k}{r}, \tag{1}$$

where μ is the reduced mass, and the equation of motion for r reads

$$\mu\ddot{r} = \mu r\dot{\theta}^2 - \frac{k}{r^2}. \tag{2}$$

For circular motion we have

$$r(t) = r_0, \qquad \ddot{r}(t) = 0 \qquad \forall t \tag{3}$$

$$\dot{\theta} = \frac{2\pi}{T}. \tag{4}$$

Pluging all this in the equation of motion (2) we get

$$r_0 = \left(\frac{kT^2}{4\pi^2\mu}\right)^{\frac{1}{3}}. \tag{5}$$

When the planets are stopped the angular velocity $\dot{\theta}$ goes to cero, and the equation of motion becomes

$$\ddot{r} = -\frac{k}{\mu}\frac{1}{r^2}. \tag{6}$$

Multiplying both sides by $2\dot{r}$ we get

$$2\dot{r}\ddot{r} = -\frac{2k}{\mu}\frac{\dot{r}}{r^2} \quad \Leftrightarrow \quad \frac{d}{dt}(\dot{r}^2) = \frac{d}{dt}\left(\frac{2k}{\mu r}\right) \quad \Leftrightarrow \quad \dot{r}^2 = \frac{2k}{\mu r} + c. \tag{7}$$

The constant c is determined by thee boundary condition, which in this case states that it must be $\dot{r} = 0$ when $r = r_0$, leading to

$$\frac{dr}{dt} = -\sqrt{\frac{2k}{\mu}} \left(\frac{1}{r} - \frac{1}{r_0} \right)^{\frac{1}{2}} = -\sqrt{\frac{2k}{\mu}} \left(\frac{r_0 - r}{r\, r_0} \right)^{\frac{1}{2}}. \tag{8}$$

We could now solve the differential equation for $r(t)$, but since we are actually interested in finding the colliding time, it is more useful to invert (8) and solve it for $t(r)$:

$$\Delta t = -\int_{r_0}^{0} \left(\frac{dt}{dr} \right) dr = -\int_{r_0}^{0} \left(\frac{dr}{dt} \right)^{-1} dr = -\sqrt{\frac{\mu}{2k}} \int_{r_0}^{0} \left(\frac{r\, r_0}{r_0 - r} \right)^{\frac{1}{2}} dr \tag{9}$$

We substitute $u = r/r_0$ and get

$$\Delta t = -\left(\frac{\mu r_0^3}{2k} \right)^{\frac{1}{2}} \int_{1}^{0} \left(\frac{u}{1-u} \right)^{\frac{1}{2}} du. \tag{10}$$

Now we change variables to $u = \sin^2 x$, $du = 2 \sin x \cos x dx$, and write the integral as

$$\Delta t = -2 \left(\frac{\mu r_0^3}{2k} \right)^{\frac{1}{2}} \int_{\pi/2}^{0} \sin^2 x dx = \left(\frac{2\mu r_0^3}{k} \right)^{\frac{1}{2}} \frac{\pi}{4}. \tag{11}$$

Substituting here with (5) we get the final result

$$\Delta t = \left(\frac{2\mu}{k} \frac{kT^2}{4\pi^2 \mu} \right)^{\frac{1}{2}} \frac{\pi}{4} = \frac{T}{4\sqrt{2}} \tag{12}$$

2. Preceding orbits

A. Since we don't add any θ dependence to the problem

$$l = \mu r^2 \dot{\theta} \tag{13}$$

is still conserved. We can therefore write the equation of motion for the one dimensional equivalent problem as

$$\mu \ddot{r} = \frac{l^2}{\mu r^3} - \frac{k}{r^2} + \frac{2c}{r^3} = \frac{l^2 + 2\mu c}{\mu r^3} - \frac{k}{r^2} \tag{14}$$

We assume that

$$\left(\frac{c\mu}{l}\right)^2 << 1 \tag{15}$$

so that we can define

$$l' = l + \frac{c\mu}{l} \tag{16}$$

and rewrite (14) as

$$\mu\ddot{r} = \frac{l'^2}{\mu r^3} - \frac{k}{r^2}. \tag{17}$$

This equation has the same form as the original Kepler equation, so we don't need to solve it again. We already know that the orbit is given by

$$r(\theta') = \frac{a(1-\varepsilon^2)}{1+\varepsilon\cos\theta'}, \tag{18}$$

where θ' is the angular velocity associated to l'. (Actually, we should have written a' and ε', but they are constants). To find the equation for the orbit as a function of the original angular variable θ we only have to find a relation between θ' and θ. Recalling that

$$l = \mu r^2\dot{\theta} \qquad\qquad l' = \mu r^2\dot{\theta}' \tag{19}$$

and

$$l' = l + \frac{c\mu}{l} \tag{20}$$

we find that

$$\mu r^2\dot{\theta}' = \mu r^2\dot{\theta}\left(1 + \frac{c\mu}{l^2}\right). \tag{21}$$

Thus, we can identify

$$\theta' = \alpha\theta \tag{22}$$

with

$$\alpha = 1 + \frac{c\mu}{l^2}, \tag{23}$$

and the equation of the orbit becomes

$$r(\theta) = \frac{a(1-\varepsilon^2)}{1+\varepsilon\cos\alpha\theta}. \tag{24}$$

B. $\alpha = 1$ corresponds to $c = 1$, which reduces the problem to the ordinary Kepler problem.

C. If $\alpha > 1$ the orbit is a preceding ellipse. To find the precession velocity we can do the following analysis. Suppose we start measuring θ at a perihelion. If there was no precession, after one year (τ) we would give a 2π turn and we would be again at a perihelion. However, given the fact that the ellipse is actually preceding, we know from (24) that the next perihelion occurs at

$$\theta_{per} = \frac{2\pi}{\alpha}. \tag{25}$$

Thus, we can calculate the precession velocity as

$$\Omega = \frac{1}{\tau}\left(\frac{2\pi}{\alpha} - 2\pi\right) = \frac{2\pi}{\tau}\left(\frac{1}{\alpha} - 1\right) \tag{26}$$