Problems 10: Hamiltonian formalism

To be handed in **ETH: Mon 14.12, UNI: Wed 16.12**

1. The harmonic oscillator revisited

a) Show that

$$\begin{aligned} Q &= q\cos\theta - \frac{p}{m\omega}\sin\theta, \\ P &= m\omega q\sin\theta + p\cos\theta, \end{aligned}$$

is a canonical transformation,
(i) by evaluating the Poisson bracket $\{Q, P\}$
(ii) by expressing $pdq - PdQ$ as an exact differential $dF_1(q, Q, t)$. Hence find the type 1 generating function of the transformation equations to express p, P in terms of q, Q.

b) Use the relation $F_2 = F_1 + PQ$ to find the type 2 generating function $F_2(q, P)$, and check your results by showing that F_2 indeed generates the transformation.

c) Assuming that the (q, p) are canonical variables for a simple harmonic oscillator with Hamiltonian

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 q^2.$$

(i) Find the Hamiltonian $K(Q, P, t)$ for the new canonical variables (Q, P), assuming that the parameter θ is some function of time. Show that we can choose $\theta(t)$ so that $K = 0$.

(ii) With this choice of $\theta(t)$ solve the new canonical equations to find the original variables (q, p) as function of time.

2. Charged particle in a uniform magnetic field

The Hamiltonian for a charged particle in a uniform magnetic field $\vec{B} = B_0 \hat{z}$ is

$$H = \frac{1}{2m}\left(\vec{p} - \frac{q}{c}\vec{A}\right)^2 ,$$

where

$$\vec{A} = \frac{B_0}{2}(-y\hat{x} + x\hat{y})$$

is the vector potential.

a) Find and solve the Hamilton equations for this system

b) Find the Hamiltonian and the Hamilton equations for the new variables q_1, q_2, p_1, p_2 defined as

$$\begin{aligned} x &= \frac{1}{\sqrt{m\omega_c}}(\sqrt{2p_1}\sin q_1 + p_2) \\ p_x &= \frac{\sqrt{m\omega_c}}{2}(\sqrt{2p_1}\cos q_1 - q_2) \\ y &= \frac{1}{\sqrt{m\omega_c}}(\sqrt{2p_1}\cos q_1 + q_2) \\ p_y &= \frac{\sqrt{m\omega_c}}{2}(-\sqrt{2p_1}\sin q_1 + p_2) \end{aligned}$$

with

$$\omega_c = \frac{B_0 q}{mc}$$

c) Solve the Hamilton equation for the canonically transformed variables, and express your result in terms of the old x, y, p_x, p_y