Sheet VIII

Due: week of November 16

Question 1 [*Linearised gravity*]:

i) In linearised gravity we write $g_{ab} = \eta_{ab} + \gamma_{ab}$. We change coordinates by defining

$$\hat{x}^a = x^a - \epsilon\xi^a(x) + \mathcal{O}(\epsilon^2) \ ,$$

where $\xi^a(x)$ is a vector field. Show that under this change of coordinates, γ_{ab} changes to

$$\hat{\gamma}_{ab} = \gamma_{ab} + \epsilon(\partial_a\xi_b + \partial_b\xi_a) + \mathcal{O}(\epsilon^2) \ .$$

ii) Calculate the Ricci tensor corresponding to $g_{ab} = \eta_{ab} + \gamma_{ab}$ to second order in γ_{ab}, starting from the definition of the Ricci tensor in terms of the metric and the Christoffel symbols,

$$R_{\mu\rho} = R_{\mu\nu\rho}{}^{\nu} = \frac{\partial}{\partial x^\nu}\Gamma^\nu_{\mu\rho} - \frac{\partial}{\partial x^\mu}\Gamma^\nu_{\nu\rho} + \Gamma^\alpha_{\mu\rho}\Gamma^\nu_{\alpha\nu} - \Gamma^\alpha_{\nu\rho}\Gamma^\nu_{\alpha\mu} \ .$$

Question 2 [*Exterior and covariant derivative*]:

The exterior derivative d of a p-form $\Omega_{\mu_1,\ldots,\mu_p}$ is defined to be the $(p+1)$-form with components

$$(d\Omega)_{\mu_0,\ldots,\mu_p} = \sum_{k=0}^{p}(-1)^k\frac{\partial}{\partial\mu_k}\,\Omega_{\mu_0\ldots\widehat{\mu_k}\ldots\mu_p} \ , \tag{1}$$

where the 'hatted' index is left out. Show that this definition is unchanged if we replace the partial derivative in (1) by the covariant derivative.

Question 3 [*Gravitational waves*]: A binary star system consists of two stars of mass M and of negligible size in a nearly Newtonian circular orbit of radius R around each other. By gravitational wave radiation the system looses energy at the rate of

$$P = \frac{1}{45}\sum_{\mu,\nu=1}^{3}\left(\frac{d^3Q_{\mu\nu}}{dt^3}\right)^2 \ ,$$

where

$$Q_{\mu\nu} = q_{\mu\nu} - \frac{1}{3}\delta_{\mu\nu}q$$

with $q = \sum_\mu q_{\mu\mu}$ and

$$q_{\mu\nu} = 3\int T^{00}x^\mu x^\nu d^3x \ .$$

[Here T^{00} is the energy density of this system in the rest frame of the binary star system.] Calculate the rate of increase of the orbital frequency due to the emission of gravitational waves. (The observation of this frequency increase by Hulse & Taylor (Nobel prize 1993) for such a binary star system is one of the best confirmations of GR.)