Sheet IV

Due: week of October 19

Question 1 [*Metric Transformations*]:

i) The metric of flat, three-dimensional Euclidean space is

$$ds^2 = dx^2 + dy^2 + dz^2\,. \tag{1}$$

Show that the metric components $g_{\mu\nu}$ in spherical polar coordinates (r, θ, ϕ) defined by

$$r = \sqrt{x^2 + y^2 + z^2}\,, \quad \cos\theta = \tfrac{z}{r}\,, \quad \tan\phi = \frac{y}{x}\,,$$

are given by

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2\,. \tag{2}$$

ii) The spacetime metric of special relativity is

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2\,. \tag{3}$$

Find the components $g_{\mu\nu}$ and $g^{\mu\nu}$ of the metric and the inverse metric, respectively, in 'rotation coordinates' defined by

$$t' = t\,, \quad x' = \sqrt{x^2 + y^2}\cos(\phi - \omega t)\,, \quad y' = \sqrt{x^2 + y^2}\sin(\phi - \omega t)\,, \quad z' = z\,, \tag{4}$$

where ω is a constant.

Question 2 [*Exterior Derivative in Action*]:

i) Let π be some p-form on $\mathbb{R}^3$. Verify by explicit calculation in the standard Euclidean basis that $d^2\pi = 0$.

ii) Let us define the one-form $\omega = xdz$. Rewrite ω in polar coordinates and find the two-form $\tilde{\omega} = d\omega$ in this basis. Show that $d^2\omega = 0$. Let $\tilde{\omega}_*$ denote the pull-back of $\tilde{\omega}$ onto the two-sphere S^2 under the natural embedding. Which value does the integral $\oint_{S^2} \tilde{\omega}_*$ take?

Hint: Stokes!

iii) Show that the one-form

$$\theta = \frac{ydx - xdy}{x^2 + y^2} \tag{5}$$

defined on $\mathbb{R}^2 \setminus \{0\}$ is closed but not exact.

Question 3 [*Interior Product*]: Let X be a vector field and Ω a p-form. We define $i_X\Omega$ to be the $(p-1)$-form given by

$$i_X\Omega(X_1, \ldots, X_{p-1}) = \Omega(X, X_1, \ldots, X_{p-1})\,. \tag{6}$$

Check the following properties:

i)
$$i_X(\Omega_1 \wedge \Omega_2) = i_X(\Omega_1) \wedge \Omega_2 + (-1)^{p_1}\Omega_1 \wedge i_X(\Omega_2)\,, \tag{7}$$
where p_1 is the degree of Ω_1.

ii) $i_X^2 = 0$.

iii) $i_X(df) = (df)(X) = X(f)$, where f is a function.

iv) $L_X = i_X \circ d + d \circ i_X$.