Sheet XII

Due: week of 14 December (or some time in the past)

Question 1 [*Time Traveling*]:

In this exercise, we want to repeat Gödel's proof of the existence of solutions to Einstein's equation that allow for timelike closed curves.

i) Let a be a constant. Consider the line element

$$ds^2 = a^2 \left(dx_0^2 - dx_1^2 + \frac{1}{2}e^{2x_1}dx_2^2 - dx_3^2 + 2e^{x_1}dx_0 dx_2 \right) . \qquad (1)$$

Find the metric components of g_{ab} and show that, for a specific choice of a and λ, this metric satisfies Einstein's equation with positive cosmological constant [1],

$$R_{ij} - \frac{1}{2}g_{ij}R = 8\pi\rho u_i u_j + \lambda g_{ij} , \qquad (2)$$

where u^i is the unit vector in the direction of x_0. Determine the relations between ρ, a and λ.

We now switch to cylindrical coordinates (t, r, ϕ) on each subspace with $x_3 \equiv$ const and define $x_3 = 2y$. It can be shown that the line element in these coordinates takes the form

$$ds^2 = 4a^2 \left(dt^2 - dr^2 - dy^2 + \left(\sinh^4 r - \sinh^2 r \right) d\phi^2 + 2\sqrt{2}\sinh^2 r\, d\phi dt \right) . \qquad (3)$$

ii) Consider the spherical curve $\mathcal{C}$ given by

$$\mathcal{C} : t = y = 0 , \quad r = R, \quad \phi \in [0, 2\pi[\qquad (4)$$

and find the set of values of R such that $\mathcal{C}$ is timelike.

iii) Now let us define the curve

$$\mathcal{C}_\alpha : y = 0 , \quad r = R, \quad t = -\alpha\phi, \quad \phi \in [0, 2\pi[, \qquad (5)$$

and argue that for sufficently small ϵ and $|\alpha| < \epsilon$ the curve $\mathcal{C}_\alpha$ is also timelike.

Let P and Q be points on the t-line and let P precede Q. With the result above, show that there exists a timelike curve that starts at Q and ends at P.

We have just proven that within Einstein's Theory of General Relativity, there exist solutions that allow for closed timelike curves. In particular, it is possible in these worlds to travel back in time or influence the past from the future.

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[1]The prefactor (here λ) to an additional term proportional to the metric in Einstein's equation is commonly called cosmological constant.