Sheet XI

Due: week of December 7

Question 1 [*Particle in Schwarzschild Background*]:

Consider a free particle moving radially in the Schwarzschild background with initial position r_0 ($r_0 \geq 2M$) and initial radial velocity $\dot{r}|_{r=r_0} = 0$. M is the mass of the Schwarzschild background.

i) Show that the particle reaches the horizon $r = 2M$ in finite proper time.

ii) How long does the particle take to get to the horizon from the point of view of a static observer at infinity?

Question 2 [*GPS*]:

The global positioning service consists of 24 satellites in circular orbits around the earth with an orbital period of 12 hours. Each satellite carries an operating atomic clock and emits timed signals that include a code telling its location. By analysing signals from at least four of these satellites, a receiver on the surface of the earth can determine its position and time.

In order for the GPS to work accurately one has to take into account effects of general relativity, in particular, the time delay between time measured in the rest frame of the satellite and time measured on earth. Assume that the satellite is in the stable circular orbit, while the observer on earth is on the equator. Determine the clock rate difference $dt_{\text{satellite}}/dt_{\text{earth}}$.

[Hint: Take into account that the orbital period of 12 hours is with respect to an observer at the corresponding radius and not with respect to an observer at infinity. The same statement holds for the orbital period of the earth's rotation.]

Question 3 [*More Schwarzschild*]:

Show that any particle behind the horizon of the Schwarzschild background, *i.e.* with $r < 2M$ must decrease its radial coordinate at a rate given by $|dr/d\tau| \geq (2M/r - 1)^{1/2}$. (This is independent of whether it is in geodesic motion or not.) Hence show that the maximum lifetime of any observer behind the horizon is $\tau = \pi M$, *i.e.* any object behind the horizon will be pulled into the singularity at $r = 0$ within this proper time. Show that this maximum time is attained by a purely radial (*i.e.* $L = 0$), free (*i.e.* geodesic) motion from $r = 2M$ with $\dot{r}|_{r=2M} = 0$.