Sheet X

Due: week of November 30

Question 1 [*Einstein-Hilbert Action*]:

i) Let us consider the Einstein-Hilbert action defined by

$$S_{\rm EH} = -\frac{1}{16\pi}\int d^4x\,\sqrt{-g}\,R\,. \tag{1}$$

Show that under the variation of $S_{\rm EH}$ with respect to the metric $g_{\mu\nu}$ the extremum condition

$$\delta S_{\rm EH} = 0 \tag{2}$$

gives the vacuum Einstein equation.

Hint: Verify that $\sqrt{-g}\,g_{\mu\nu}\delta R^{\mu\nu}$ can be written as $\partial_\mu(\sqrt{-g}\,V^\mu)$ with V^μ vanishing on the boundary. The derivative of the determinant g was derived on Sheet VI.

ii) Verify that the electromagnetic energy-stress-tensor $T_{\rm em}$ can be written as

$$T^{\mu\nu}_{\rm em}(x) = \frac{2}{\sqrt{-g}}\frac{\delta S_{\rm em}}{\delta g_{\mu\nu}}(x)\,, \tag{3}$$

where

$$S_{\rm em} = -\frac{1}{16\pi}\int d^4x\,\sqrt{-g}F^{\mu\nu}(x)F_{\mu\nu}(x)\,. \tag{4}$$

iii) In general, given any action for matter fields $S_{\rm matter}$, its energy-stress-tensor may be found by varying with respect to the metric,

$$T^{\mu\nu}_{\rm matter} = \frac{2}{\sqrt{-g}}\frac{\delta S_{\rm matter}}{\delta g_{\mu\nu}}\,. \tag{5}$$

From this, determine Einstein's equation including matter degrees of freedom.

Question 2 [*Singularities*]:

In the lecture, the Schwarzschild solution to Einstein's equation in the vacuum was introduced:

$$ds^2 = -\left(1-\frac{2m}{r}\right)dt^2 + \frac{dr^2}{1-\frac{2m}{r}} + r^2 d\Omega\,. \tag{6}$$

i) Determine the curvature scalar $\mathcal{R} = R_{abcd}R^{abcd}$.

Hint: The only non-vanishing components of the Riemann tensor are ${R^t}_{rrt}$, ${R^t}_{\phi\phi t}$, ${R^t}_{\theta\theta t}$, ${R^r}_{\phi\phi r}$, ${R^r}_{\theta\theta r}$ and ${R^\phi}_{\theta\theta\phi}$.

ii) By checking the behavior of $\mathcal{R}$ at $r = 2m$ ($\mathcal{H}$) and $r = 0$ ($\mathcal{S}$), argue that $\mathcal{H}$ is coordinate singularity while $\mathcal{S}$ is a proper singularity, i.e., show that $\mathcal{R}$ diverges at $r = 0$ but not at $r = 2m$.