Sheet I

Due: week of September 28

Notation: In this lecture we use natural units, that is, we are setting $c = G = 1$.

Question 1 [*Addition Law for Velocities*]:

The Lorentz transformation law mapping an intertial system into one moving with the velocity v relative to the first is of the form

$$\hat{t} = \gamma(t - vx) \quad \hat{x} = \gamma(x - vt) \quad \hat{y} = y \quad \hat{z} = z\,,$$

where $\gamma = 1/\sqrt{1-v^2}$.

i) Show that the successive application of two such Lorentz transformations, $(x,t) \xrightarrow{v_1} (\hat{x},\hat{t})$ and $(\hat{x},\hat{t}) \xrightarrow{v_2} (\tilde{x},\tilde{t})$, where only the x-component of v_1 and v_2 is non-zero, is a Lorentz transformation. Identify the velocity of the resulting Lorentz transformation.

ii) Suppose that only the x-component of v_1 and the y-component of v_2 are non-zero. Show that the successive application of two such boosts can be written as the successive application of a boost (with respect to a rotated coordinate system), and a rotation. Find the direction and the absolute value of the velocity of the resulting boost.

Question 2 [*Lorentz Transformations for Acceleration*]:

Suppose a coordinate system K' is moving with a velocity $\mathbf{v}$ with respect to another coordinate system K. A particle in K' is moving with a velocity $\mathbf{u}'$ and is subject to a constant acceleration $\mathbf{a}'$.

Determine the Lorentz transformation law for accelerations by showing that the components of the acceleration parallel and perpendicular to the relative velocity $\mathbf{v}$ in K are given by

$$\mathbf{a}_\parallel = \frac{(1-v^2)^{3/2}}{(1+\mathbf{v}\cdot\mathbf{u}')^3}\mathbf{a}'_\parallel\,, \qquad \mathbf{a}_\perp = \frac{(1-v^2)}{(1+\mathbf{v}\cdot\mathbf{u}')^3}\left(\mathbf{a}'_\perp + \mathbf{v}\times(\mathbf{a}'\times\mathbf{u}')\right)\,.$$

Remark: In this context, velocity and acceleration of a particle means the first and second derivative of the trajectory of the particle with respect to the *coordinate time* and not its eigentime.

Question 3 [*Car Paradox*]:

A car enters a 3m long garage with a very high velocity that corresponds to $\gamma = 2$. The length of the car at rest is known to be 4m. The garage has two gates at each end of it, which are open. In the perception of a gate keeper in the rest frame of the garage, the length of the car is just 2m, so he concludes that it fits perfectly into the garage. However, in the rest frame of the car, the length of the garage is just 1.5m and therefore the driver of the car will conclude that the car does not fit in the garage. Explain why the different points of view do not create a paradox.

From now on suppose that the gate at the end of the garage is closed. From the point of view of the gate keeper, the car has a braking distance of 1m after it completely entered the garage. Therefore a braking system will be activated right after the car completely entered the garage and simultaneously the front gate will be closed. Then the whole car will be in the closed garage. Confusion arises from the point of view of the driver since from his perspective the car projects 2.5m beyond the garage when the braking system is activated. He is afraid that the front gate might damage the rear of his car when it is closed. Why is this fear without any reason?
However, he still needs to be afraid of his car being damaged since obviously his car will be compressed to 3m during the braking process. How can that happen?