Quantum Field Theory I, Exercise Set 9.

HS 08 *Due:* 27/28 November 2008

1. QED Lagrangian

The QED Lagrangian density is given by

$$\mathcal{L} = \frac{1}{4e^2} F_{\mu\nu}F^{\mu\nu} + \mathrm{i}\overline{\psi}\gamma^\mu\nabla_\mu\psi - m\overline{\psi}\psi - \frac{\tilde{\mu}^2}{2}A_\mu A^\mu \,, \tag{1}$$

where $\nabla_\mu = \partial_\mu - \mathrm{i}A_\mu$ is the covariant derivative on the space of spinor fields.

(i) What are the equations of motion in the Lorentz gauge?

(ii) A gauge transformation is defined by

$$A^\mu \to A^\mu + \partial^\mu\chi \,, \quad \overline{\psi} \to \overline{\psi}\mathrm{e}^{-\mathrm{i}\chi} \,, \quad \psi \to \mathrm{e}^{\mathrm{i}\chi}\psi \,.$$

Show that, for $\tilde{\mu}^2 = 0$, the Lagrangian density (1) is invariant under such transformations. Show that the conserved current $j^\mu = \overline{\psi}\gamma^\mu\psi$ is invariant as well.

2. Furry's theorem

(i) Using Wick's theorem compute

$$\langle 0|\mathrm{T}\left(\psi_{\alpha_1}(x_1)\overline{\psi}_{\beta_1}(y_1)\ldots\psi_{\alpha_n}(x_n)\overline{\psi}_{\beta_n}(y_n)\right)|0\rangle \,.$$

(ii) Show that each fermion loop carries a factor (-1).

(iii) Verify that

$$\mathrm{C}S_F(x-y)\mathrm{C}^{-1} = S_F^\top(y-x) \,,$$

where $\mathrm{C} = i\gamma^2\gamma^0$ is charge conjugation.

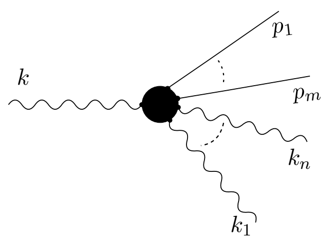
(vi) Prove Furry's theorem using the above results.

3. Gauge invariance and polarisation sum for photons

Consider an external photon with polarisation $\varepsilon(k)$ interacting with n external photons (momenta $k_1, \ldots, k_n$) and m external fermions (momenta $p_1, \ldots, p_m$). We write the invariant transition amplitude in the following way

$$\mathcal{M}(k) = \varepsilon^\alpha(k)\mathcal{M}_\alpha(k) \,, \tag{2}$$

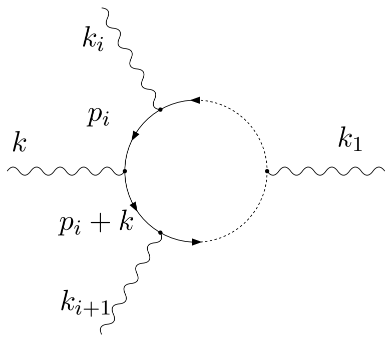
where dependences on the other momenta and polarisations are implicit.



(i) Under a gauge transformation $A^\alpha(x) \to A^\alpha(x) + \partial^\alpha\chi(x)$, the polarisation vector $\varepsilon^\alpha(k)$ transforms as $\varepsilon_r^\alpha(k)e^{\pm ik\cdot x} \to [\varepsilon_r^\alpha(k) \pm i\hat{\chi}(k)k^\alpha]e^{\pm ik\cdot x}$. Convince yourself, that by gauge invariance the amplitude has to vanish if we replace $\varepsilon^\alpha(k)$ by k^α in (2), i.e.,

$$k^\alpha \mathcal{M}_\alpha(k) = 0 \; . \tag{3}$$

(ii) Prove (3) for the case where the photon, whose polarisation vector is replaced by its momentum, couples to a closed fermion loop with another n external photons attached.



(iii) Verify the polarisation sum for photons

$$\sum_{r=1}^{2} \varepsilon_r^\alpha(k)\varepsilon_r^\beta(k) = -g^{\alpha\beta} - \frac{k^\alpha k^\beta - (k\cdot n)(k^\alpha n^\beta + n^\alpha k^\beta)}{(k\cdot n)^2} \; ,$$

where n^α is a timelike vector ($n^2 = 1$).

(iv) Show that

$$\sum_{r=1}^{2} |\varepsilon_r^\alpha(k)\mathcal{M}_\alpha(k_1, k_2, \ldots, k, \ldots, k_n)|^2 = -\mathcal{M}_\alpha(k_1, k_2, \ldots, k, \ldots, k_n)\overline{\mathcal{M}^\alpha(k_1, k_2, \ldots, k, \ldots, k_n)} \; .$$

4. Electron-positron scattering in QED

Draw all Feynman graphs in momentum space which contribute to $e^-e^+ \to e^-e^+$ scattering in 2[nd] order in e_R, the renormalised electron charge.