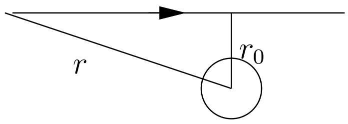
General relativity, exercise sheet 11.

HS 08 *Due:* Fri, December 12, 2008

1. Time delay in the Schwarzschild metric

Consider a ray passing near the Sun at minimal distance r_0.



Non relativistically it takes light a time $t = \sqrt{r^2 - r_0^2}$, $(c = 1)$ to reach radius r_0 from r (or vice versa).

i) Show that in Schwarzschild coordinates this time is

$$t = \int_{r_0}^{r} \frac{dr}{1 - \frac{2m}{r}} \left(1 - \frac{1 - 2m/r}{1 - 2m/r_0} \left(\frac{r_0}{r} \right)^2 \right)^{-1/2} . \tag{1}$$

Hint: Use the radial eq. $\dot{r}^2 + V(r) = \mathcal{E}^2$ and express $\dot{r} = dr/d\tau$ by dr/dt using the conservation of $\mathcal{E}$. Establish a relation between $l/\mathcal{E}$ and r_0.

ii) Compute (1) for small m/r_0 and conclude that the Shapiro time delay $\Delta t = t - \sqrt{r^2 - r_0^2}$ is

$$\Delta t(r) = 2m \log \left(\frac{r + \sqrt{r^2 - r_0^2}}{r_0} \right) + m \left(\frac{r - r_0}{r + r_0} \right)^{1/2} + O(m^2) .$$

iii) Let the ray join two planets, e.g. Earth and Venus, at radii r_1 and r_2 on opposite sides of r_0. The round trip delay,

$$\Delta t = 2(\Delta t(r_1) + \Delta t(r_2)) ,$$

of a radar signal is measurable. Compute it for $r_1, r_2 \gg r_0$.

2. Radial free fall

i) Find the motion $r(\tau)$ of a particle falling radially inward from $r = R$ towards a black hole and starting from rest in Schwarzschild coordinates. Note that $r(\tau)$ can not be expressed in closed form, but there is a parametric representation $r = r(\eta)$, $\tau = \tau(\eta)$ which can.

Hint: The radial equation has been encountered before in another context.