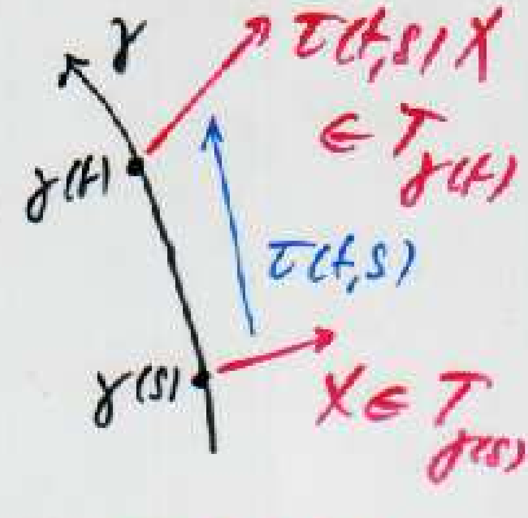
Definition: Along any curve γ in M a parallel transport of vectors is defined: a linear map



$$\tau(t,s) : T_{\gamma(s)} \to T_{\gamma(t)}, \quad X \mapsto \tau(t,s)X$$

with

1) $\tau(t,t) = 1$, $\quad \tau(t,s)\tau(s,r) = \tau(t,r)$

2) in any chart

$$\frac{\partial}{\partial t} \tau^i{}_k(t,s)\Big|_{t=s} = -\Gamma^i{}_{\ell k}(\gamma(s))\dot{\gamma}^\ell(s)$$

$\Gamma^i{}_{\ell k}(x)$: Christoffel symbols of the transports

Properties

- a parallel transported vector

$$X(t) = \tau(t,s)X(s)$$

obeys the ODE

$$\dot{X}^i(t) + \Gamma^i{}_{\ell k}(\gamma(t))\dot{\gamma}^\ell(t)X^k(t) = 0 \, ;$$

and viceversa.

$\dot{X}^i$ are derivatives w.r.t. t of components $X^i(t)$ of a vector $X(t) \in T_{\gamma(t)}$; but <u>not</u> themselves the components of a vector.

→ $\Gamma^i_{\ell k}$ are not the components of a tensor

- $\frac{\partial}{\partial s} \tau^i_j(t,s)\Big|_{s=t} = \Gamma^i_{\ell j}(\gamma(t))\dot{\gamma}^\ell(t)$

- change of coordinates $x \leftrightarrow \bar{x}$

$$\bar{\Gamma}^i_{\ell k} = \Gamma^p_{rq} \frac{\partial \bar{x}^i}{\partial x^p} \frac{\partial x^q}{\partial \bar{x}^k} \frac{\partial x^r}{\partial \bar{x}^\ell} + \frac{\partial \bar{x}^i}{\partial x^p} \frac{\partial^2 x^p}{\partial \bar{x}^k \partial \bar{x}^\ell}$$

- Properties of $\nabla_X Y$ (X, Y vector fields)

(i) $\nabla_X Y$ is a vector field, linear in X, Y.

(ii) $\nabla_{fX} Y = f \nabla_X Y$ (f-linear in X)

(iii) $\nabla_X (fY) = f \nabla_X Y + (Xf) Y$

- Def. ∇ with (i–iii) is an affine connection

- Components w.r.t. coordinate basis $e_i = \frac{\partial}{\partial x^i}$

$$(\nabla_X Y)^i = (Y^i{}_{,l} + \Gamma^i{}_{lk} Y^k) X^l$$

- Affine connections and parallel transports determine each other bijectively through:

$$* \qquad \nabla_{e_l} e_k = \Gamma^i{}_{lk} e_i$$

or, equivalently,

* let $\dot\gamma(s) = X_{\gamma(s)}$ (i.e. γ is orbit of X) and $Y(s) \in T_{\gamma(s)}$ a vector field on γ.

Then

$$Y(s) = \tau(s,t) Y(t) \quad \text{iff} \quad \nabla_X Y = 0$$

"Y is parallel trsp."

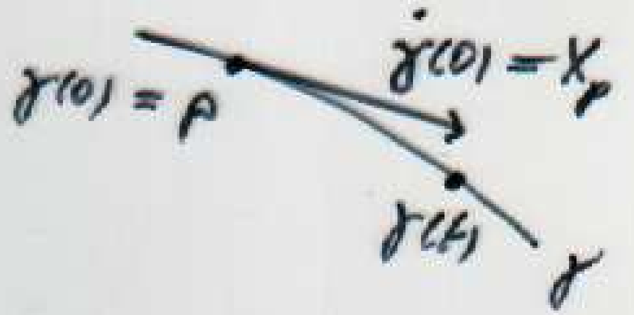
<u>Definition</u> of covariant derivative ∇_X (associated to parallel transport τ)

X: vector field

R: tensor "

$$(\nabla_X R)_p = \frac{d}{dt}\, \tau(0,t) R_{\gamma(t)} \Big|_{t=0}$$

where: $\gamma(t)$ curve through $p = \gamma(0)$
with $\dot\gamma(0) = X_p$



<u>Remark</u> $(\nabla_X R)_p$ is defined for $p \in \gamma$ as soon as R_p is for $p \in \gamma$ ($p \in M$ not needed)

Torsion and curvature

∇ affine connection on M ; X, Y vector fields

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y] \qquad \text{(torsion)}$$

$$R(X,Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]} \qquad \text{(curvature)}$$

- At first: $T(X,Y)$ vector field

  $R(X,Y)$ linear self-map on tensor fields (preserving type)

- $T(X,Y) = -T(X,Y)$, $R(X,Y) = -R(Y,X)$

- $$\langle \omega, T(X,Y) \rangle =: T^k{}_{ij}\, \omega_k X^i Y^j$$

  $$\langle \omega, R(X,Y) Z \rangle =: R^i{}_{jkl}\, \omega_i Z^j X^k Y^l$$

are tensor fields of type $\binom{1}{2}$, resp. $\binom{1}{3}$

- properties

  $R(X,Y) f = 0$

  $R(X,Y)(S \otimes T) = (R(X,Y)S) \otimes T + S \otimes (R(X,Y)T)$

  $\operatorname{tr} R(X,Y) T = R(X,Y) \operatorname{tr} T$

  $\langle \omega, R(X,Y) Z \rangle = -\langle R(X,Y)\omega, Z \rangle$

- components

$$T^k_{ij} = \Gamma^k_{ij} - \Gamma^k_{ji}$$

$$R^i_{jkl} = \Gamma^i_{lj,k} - \Gamma^i_{kj,l} + \Gamma^s_{lj}\Gamma^i_{ks} - \Gamma^s_{kj}\Gamma^i_{ls}$$

- Bianchi identities

1) $R(X,Y)Z$ + cyclic perm. $= 0$

2) $(\nabla_X R)(Y,Z) +$ " $= 0$

Def. A pseudo-Riemannian metric on M is a • symmetric • non-degenerate tensor field g of type $\binom{0}{2}$, i.e.

- $g(X,Y) = g(Y,X) \equiv (X,Y)$
- $g_p(X_p, Y_p) = 0$, for all $Y_p \in T_p \Rightarrow X_p = 0$

(X, Y vector fields ; $X_p, Y_p \in T_p$)

g_p is called inner product on T_p

Aside: Riemannian metric : "non-degenerate" replaced by the stronger property "positive definite":

- $g_p(X_p, X_p) \geq 0$ with $= 0$ iff $X_p = 0$.

Lowering the index:

g: vector fields $\rightarrow$ 1-forms

$\qquad X \quad \mapsto gX = \tilde{X}$

- defined by $\langle gX, Y \rangle = (X, Y)$
- reads as $\tilde{X}_i = g_{ik} X^k$ (drop $\sim$)

Theorem (M, g) pseudo-Riemannian manifold. Then there is a unique affine connection ∇ with

- torsion $T = 0$
- $\nabla g = 0$

It is given as

$$2(\nabla_X Y, Z) = X(Y, Z) + Y(Z, X) - Z(X, Y)$$
$$- ([Y, Z], X) + ([Z, X], Y) + ([X, Y], Z)$$

Its Christoffel symbols are

$$\Gamma^i{}_{jk} = \frac{1}{2} g^{il} (g_{lj,k} + g_{kl,j} - g_{jk,l})$$

The connection is known as Riemann (or Levi-Civita) connection.

<u>Def</u>. A parameterized curve $x(\lambda)$ in M is a <u>geodesic</u> if it solves the variational problem

$$\delta \int_{(1)}^{(2)} d\lambda \; (\dot{x}, \dot{x}) = 0$$

($\dot{} = d/d\lambda$) with fixed endpoints λ_i, $x(\lambda_i)$ ($i = 1, 2$).

The corresponding Euler-Lagrange eqs. are

$$\ddot{x}^i + \Gamma^i_{\ell k}(x)\, \dot{x}^\ell \dot{x}^k = 0$$

i.e. $\nabla_{\dot{x}} \dot{x} = 0$: tangent vector is parallel transport in its own direction

Remarks:

1) $L(x, \dot{x}) = (\dot{x}, \dot{x})$

is conserved along the geodesic

2) Admissible reparametrizations:

$$\lambda \longmapsto \lambda' = a\lambda + b$$

(a, b fixed)

M pseudo-Riemannian manifold, metric g

<u>Theorem</u> In some neighborhood of any point $p \in M$ there is a chart such that

(i) $x^i = 0$ at p

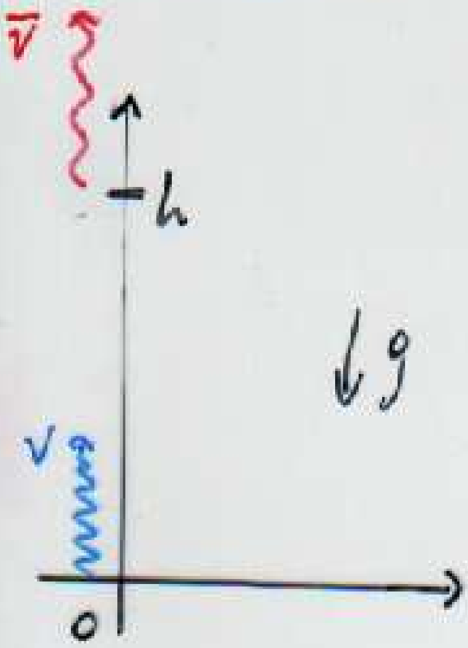
(ii) $g_{ij}(0) = \eta_{ij}$, $(\eta = \mathrm{diag}(\pm 1, \dots \pm 1))$

(iii) $g_{ij,l}(0) = 0$ $\quad (\Leftrightarrow \Gamma^i_{lj}(0) = 0)$

x^i are called <u>normal coordinates</u> at p.

Application of the equivalence principle

Gravitational red-shift



$$\bar{\nu} = \nu \left(1 - \frac{gh}{c^2}\right)$$

Newtonian limit of free fall gives:

$$g_{00} = 1 + \frac{2\varphi}{c^2}$$

φ: Newtonian gravitational potential

The equivalence principle (EP)

"All freely falling, non-rotating local reference frames ("local inertial frames", LIF) are equivalent regarding all local experiments".

The postulates of General Relativity (GR)

1. Time and space form a 4-dim. pseudo-Riem. manifold M, with metric g of signature $(1,-1,-1,-1)$. The metric expresses measurements by means of ideal clocks and rods.
2. Physical laws are equations among tensors (general covariance)
3. With the exception of g, laws contain only quantities present in Special Relativity (SR).
4. In a LIF ($\rightarrow$ normal coordinates) laws are as in SR.

Clocks:

Let $(x^0, .. x^3)$ be coordinates such that world line of clock is

$$x = (ct, 0, 0, 0)$$

Then, a proper time interval $\Delta\tau$ of the clock is

$$(\Delta\tau)^2 = g_{00}(x)\,(\Delta t)^2.$$

The Global Positioning System

- 24 satellites orbiting around the Earth

  radius ~ 26'000 km

  inclination to the equatorial plane $\approx 55^\circ$

  ≥ 4 of them visible at any moment from anywhere on Earth.

- If metric is Minkowski, then in any inertial frame (in the sense of SR)

$$|\vec{x} - \vec{x}_i(t_i)| = c|t-t_i| \qquad i=1,2,3,4$$

$(c = 30\ \mathrm{cm/ns})$

where

$(t, \vec{x})$: receiving event of all 4 signals

$$\left.\begin{array}{l} t_i \text{ emission time} \\ \vec{x}_i(t) \text{ orbit} \end{array}\right\} \text{of the } i\text{-th satellite (known!)}$$

$\rightarrow$ the 4 unknowns $(t, \vec{x})$ are determined!

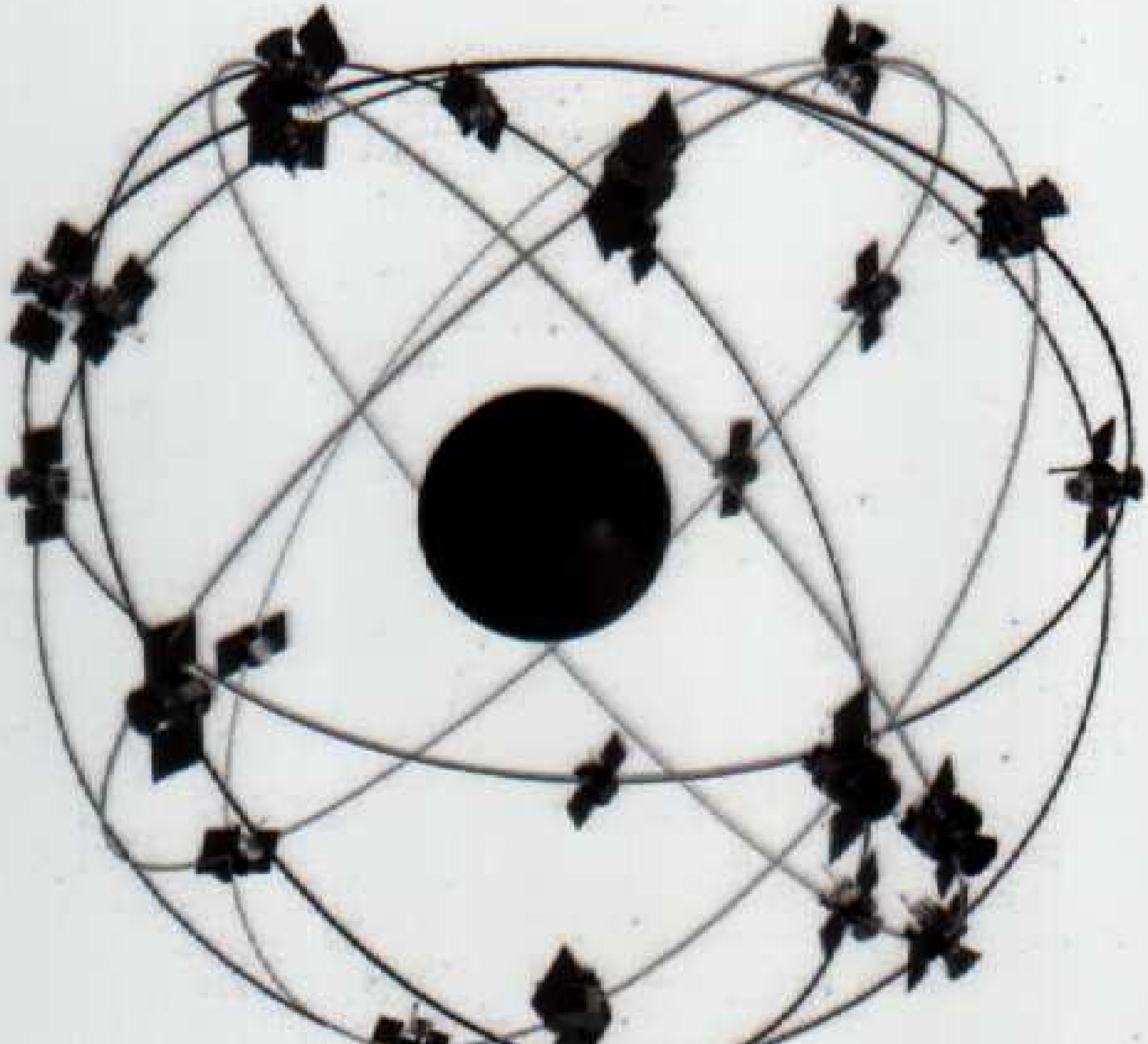
GPS : precision 18 ns or 6 m

[6 ns " 2m clock prec

companies have GPS-equipped vehicles that give directions to drivers on display screens and through synthesized voice instructions. Imagine never again getting lost on vacation, no matter where you are.

GPS-equipped balloons monitor holes in the ozone layer over the polar regions as well as air quality across the nation. Buoys tracking major oil spills transmit data using GPS to guide cleanup operations. Archaeologists, biologists, and explorers are using the system to locate ancient ruins, migrating animal herds, and endangered species such as manatees, snow leopards, and giant pandas. GPS is also an important tool used by farmers in agriculture and herding.

The future of GPS is as unlimited as your imagination. New applications will continue to be created as technology evolves. GPS satellites, like stars in the sky, will be guiding us well into the 21st century.



Twenty-four GPS satellites orbit 11,000 nautical miles above Earth to serve military and civilian users around the clock. This network of satellites forms the core of the most precise navigation system ever invented.

- Metric in local inertial frame centered at the center of the Earth (radial coords.)

$$(ds)^2 = c^2\left(1+\frac{2\varphi}{c^2}\right)(dt)^2 - \left[\left(1-\frac{2\varphi}{c^2}\right)dr^2 + r^2 d\Omega^2\right]$$

metric on 2-sphere

$$\varphi = -\frac{GM}{r}$$

G : Newton's constant

- time $\leftrightarrow$ space coordinates for light propagation $ds = 0$. E.g. for satellite (radius r_2) above receiver (radius r_1)

$$dr = c\,dt \cdot \sqrt{\frac{1+\frac{2\varphi}{c^2}}{1-\frac{2\varphi}{c^2}}} = c\,dt\left(1+\frac{2\varphi}{c^2}+\ldots\right)$$

$$\Delta r = c\,\Delta t - \frac{2GM}{c}\int_{r_2}^{r_1}\frac{dt}{r}$$

$$= c\,\Delta t + \underbrace{\frac{2GM}{c^2}\log\frac{r_2}{r_1}}_{\text{Shapiro delay } \approx 2\,\text{cm}}$$

Not (yet) relevant.

But: how to construct "coordinate time" on a device?

Clock: velocity v, radius r

$$c^2 d\tau^2 = c^2\left(1+\frac{2\varphi}{c^2}\right)dt^2 - \vec{v}^2 dt^2$$

$$d\tau = \sqrt{1-\frac{v^2}{c^2}+\frac{2\varphi}{c^2}}\, dt$$

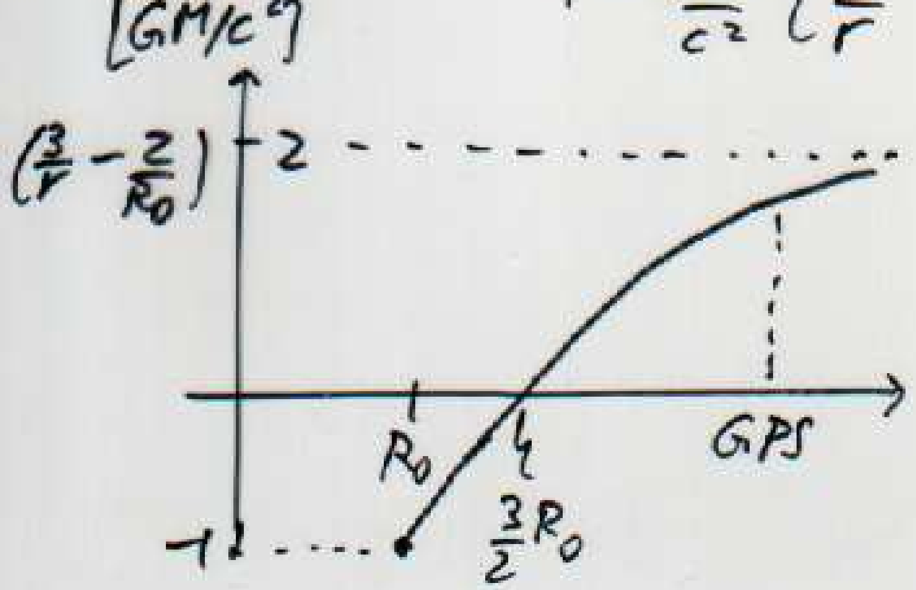
$$\underset{\text{SR}}{\uparrow} \qquad \underset{\text{GR}}{\uparrow}$$

Satellite: $\dfrac{\vec{v}^2}{r} = \dfrac{GM}{r^2}$

$$\varphi = -\frac{GM}{r}\Big|_{R_0}^{r} \quad : \; R_0 : \text{radius of Earth}$$

$$1-\frac{\vec{v}^2}{c^2}+\frac{2\varphi}{c^2} = 1+\frac{GM}{c^2}\left(-\frac{1}{r}-\frac{2}{r}+\frac{2}{R_0}\right)$$

$$= 1-\frac{GM}{c^2}\left(\frac{3}{r}-\frac{2}{R_0}\right)$$



For GPS:

$$\frac{d\tau}{dt} = 1 + 4.5 \cdot 10^{-10}$$

Needs to be taken into account for accurate position determination

Energy and momentum in SR

- Particle : Energy-momentum vector

$$p^\mu = \left(\frac{E}{c}, \vec{p}\right) = \frac{m}{\sqrt{1-\frac{v^2}{c^2}}} \, (c, \vec{v})$$

(m : (rest) mass)

E: energy

p^i: momentum (i-th component)

- Field : Energy-momentum tensor $T^{\mu\nu}$

T^{00}: energy density ; T^{0k}: $\frac{1}{c}\cdot$ energy current density (k direction)

T^{i0}: $c\cdot$ momentum density; (i-th component) T^{ik} = momentum current density (i-th comp, k direction)

The Einstein field equations

$$G_{\mu\nu} = \varkappa T_{\mu\nu}$$

where

$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}$ Einstein tensor

$T_{\mu\nu}$ total energy-momentum tensor

$\varkappa = \frac{8\pi G_0}{c^4}$ gravitational constant

They are 10 non-linear 2nd order partial differential equations for $g_{\mu\nu}$ with integrability condition

$$T^{\mu\nu}{}_{;\nu} = 0$$

Alternate form

$$R_{\mu\nu} = \varkappa \left(T_{\mu\nu} - \frac{1}{2} T g_{\mu\nu} \right)$$

special cases:

- $T \equiv T^{\mu}{}_{\mu} = 0$: $R_{\mu\nu} = \varkappa T_{\mu\nu}$
- $T = 0$ (vacuum) : $R_{\mu\nu} = 0$

$u(x)$: geodesic vector field of 4-velocities

$$\nabla_u u = 0 \quad , \quad (u,u) = c^2$$

$n(x)$: displacement field, Lie transported by u :

$$[n, u] = 0$$

Then

$$\nabla_u^2 n = R(u,n)\, u$$

(equation of geodesic deviation) : describes relative acceleration of nearby $\widehat{\text{test}}$ particles.

On average :

$$\sum_{i=1}^{3} \langle e^i, \nabla_u^2 e_i \rangle = -R_{\mu\nu} u^\mu u^\nu$$

For ideal fluid

$$T^{\mu\nu} = \left(\rho + \frac{p}{c^2}\right) u^\mu u^\nu - p g^{\mu\nu} :$$

$$\sum_{i=1}^{3} \langle e^i, \nabla_u^2 e_i \rangle = -\frac{\varkappa c^2}{2} (\rho c^2 + 3p)$$

$$< 0 \quad \text{if} \quad \rho c^2 + 3p > 0 .$$