General Relativity

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ETH Zürich, HS2008

Lecture Notes

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1 Manifolds and Tensorfields

1.1 Differentiable Manifold

A differentiable Manifold is definded by the following elements:
In the overlap between any 2 charts the change of coordinates is smooth.

$$\dim\ M = n$$

<u>**Concepts:**</u> **(definded through the charts)**

- differentiable function $f : M \longrightarrow \mathbb{R}$
  i.e.
  $f(p(x)) = f(x)$ is differentiable as a map $K \longrightarrow \mathbb{R}^n$ (algebra $\mathcal{F}$(M): mult. & add.)
- $\mathcal{F}_p$: algebra of smooth function defined in an arbitrary small neighbourhood of $p \in M$
  $f = g$ if $f(p') = g(p')$ in some intersection p' of p
- differentiable curves: $\gamma : \mathbb{R} \longrightarrow M$
- differentiable maps: $M \longrightarrow M'$

Tangent Space: T_p **at** $p \in M$

<u>Definition:</u> A vector $X \in T_p$ is a "derivation"& linear map

$$X : \mathcal{F}_p \longrightarrow \mathbb{R}$$

with a product rule

$$X(fg) = (Xf)g(p) + f(p)(Xg)$$

In any chart $K \ni p$

$$Xf = X^i f_{,i}(x)$$

where: $,i = \frac{\partial}{\partial x_i}$ and
$X^i = X(x_i)$
$X^i : M \to \mathbb{R}$ coordinate functions

<u>Proof:</u> $f \equiv I \Rightarrow f = f^2$, $Xf = X(ff) = 2Xf$, $Xf = 0$
same for $f \equiv const$ suppose $p \to x = 0$

$$f(x) = f(0) + \int_0^1 \frac{d}{dt} f(tx)\, dt$$

$$= f(0) + \underbrace{x^i \int_0^1 f_{,i}(tx)\, dt}_{g_i(x)}$$

$$\Rightarrow \quad Xf = 0 + (Xx^i) g_i(0) + \underbrace{x^i\,|_{x=0}}_{=0}(Xg^i)$$

In particular, in any chart:
$\frac{\partial}{\partial x^i}: \quad \longrightarrow f_{,i}(x)$ is a derivaion $\frac{\partial}{\partial x_i} \in T_p$

$\Rightarrow X\not{f} = x^i \frac{\partial}{\partial x^i}\not{f}$ holds for any f
$\left(\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}\right)$ is a basis of T_p: canonical basis
$(\rightarrow \dim T_p = n)$

Directional Derivatives:

Let $\underset{\gamma(t)\in M}{\gamma}$ be a curve through $\gamma(0) = p$
γ defines $X \in T_p$ by

$$Xf = \frac{d}{dt} f(\gamma(t)) \,|_{t=0} \qquad f \in \mathcal{F}_p$$

In components:

$$Xf = \frac{\partial f}{\partial x^i}\frac{\partial \gamma}{\partial t} \,|_{t=0} \quad \rightarrow \quad x^i = \frac{d\gamma^i}{dt} \,|_{t=0}$$

Thus: $X \in T_p$ $\in$ equiv. classes of "tangent vectors"to curves through p.

Bases of T_p

with respect to bases $(e_1, \ldots, e_n)$:

$$X = X^i e_i$$

Change of basis:

$$\bar{e}_i = \phi_i{}^k e_k \qquad \bar{X}^i = \phi^i{}_k X^k$$

$$\Rightarrow \quad X = \bar{X}^i \bar{e}_i = \underbrace{\phi^i{}_k \phi_i{}^l}_{\substack{\delta_{kl} \text{ since valid for every } e_i \\ \Rightarrow \phi^i{}_k \phi_i{}^l X^k = X^l}} X^k e_l = X^l e_l$$

$$\Rightarrow \phi^i{}_k = \left(\phi^{-1}\right)^{T\ k}_{\ \ i}$$

special case: $e_i = \frac{\partial}{\partial x_i}$ canonical basis of K
$\bar{e}_i = \frac{\partial}{\partial \bar{x}_i}$ canonical basis of $\bar{K}$

<u>change of coordinates:</u> $x \Leftrightarrow \bar{x}$

$$\bar{e}_i = \tfrac{\partial}{\partial \bar{x}_i} = \tfrac{\partial x^k}{\partial \bar{x}^i} \underbrace{\frac{\partial}{\partial x^k}}_{e_k} \quad \Rightarrow \quad {\phi_i}^k = \tfrac{\partial x^k}{\partial \bar{x}^i}$$

$$\Updownarrow$$

$$\bar{X}^i = X(\bar{x}^i) = x^k \tfrac{\partial \bar{x}^i}{\partial x^k} \quad \Rightarrow \quad {\phi^i}_k = \tfrac{\partial \bar{x}^i}{\partial x^k}$$

$\Big($ Comparison with a "phyicist"definition of vectors: set of components $(X_i)^n$ $\Big)$

Cotangent Space: ${T_p}^*$ **: dual linear space of** T_p

Def.: Covector: $\omega \in {T_p}^*$ is a linear form

$\omega : T_p \to \mathbb{R}$

$X \longmapsto \omega(X) \equiv \langle \omega, X \rangle$ "duality bracket"

$({T_p}^{**} \cong T_p)$

Basis $(e^1, \dots, e^n)$ of ${T_p}^* \Leftrightarrow \omega = \omega_i e^i$ with components of the covector ω_i

In particular: dual basis to $(e_1, \dots, e_n)$ of T_p:

$$\left\langle e^i, X \right\rangle = X^i$$
$$\left\langle e^i, e_j \right\rangle = \delta_{ij}$$

Let $f \in \mathcal{F}_p$:

$$df : X \longrightarrow df(X) := Xf \quad df \in {T_p}^*$$

Components:

$$(df)(X) = \underbrace{X^i}_{X(x^i)} f_{,i}(x) = f_{,i}(x)(dx^i)(X)$$

$$(df) = f_{,i}(x)(dx^i)$$

$\Rightarrow (dx^1, \dots, dx^n)$ is a basis of ${T_p}^* : \langle dx^i, X \rangle = X^i$

- is the dual basis to $\left(\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}\right)$
- all $\omega \in {T_p}^*$ are of the form

$$\omega = df \qquad \text{for some } f \in T_p \qquad \text{(pointwise, not really)}$$

change of coordinates:

$$\langle d\bar{x}^i, X \rangle = \bar{X}^i = \tfrac{\partial \bar{x}^i}{\partial x^k} \underbrace{X^k}_{\langle dx^k, X \rangle} \quad \Rightarrow d\bar{x}^i = \tfrac{\partial \bar{x}^i}{\partial x^k} dx^k$$

Tensors on T_p:

tensors of type $\begin{pmatrix} p \\ q \end{pmatrix} \Leftrightarrow \begin{cases} \text{p times contravariant} \\ \text{q times covariant} \end{cases}$

$T(\omega, X, Y)$ is a trilinear form on ${T_p}^* \times T_p \times T_p$

$\Rightarrow$ generalizes vectors, covectors

- tensor product

$$T(\omega, X, Y) = R(\omega, X) \cdot S(Y): \quad T = R \otimes S$$

- components, e.g. T of type $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$$T(\underbrace{\omega}_{\omega_i e^i}, \underbrace{X}_{x^j e_j}, Y) = \underbrace{T\left(e^i, e^j, e^k\right)}_{\substack{{T^i}_{jk} \\ \text{components}}} \qquad \underbrace{\omega X^j Y^k}_{\substack{e_i(\omega) e^j(X) e^k(Y) \\ =(e_i \otimes e^j \otimes e^k)(\omega, X, Y)}}$$

$$\Rightarrow \quad T = {T^i}_{jk} e_i \otimes e^j \otimes e^k$$

$$\left\{ \text{Tensor of type} \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\} = \left\{ \text{lin. comb. of tensor product} X \otimes \omega \otimes \omega' \right\}$$
$$=: T_p \otimes {T_p}^* \otimes {T_p}^*$$

<u>change of coordinates:</u> $\left(\bar{e}_i = {\phi_i}^\alpha e_\alpha, \quad \bar{e}^i = {\phi^i}_\beta e^\beta\right)$

$$\bar{T}^i_{jk} = T\left(\bar{e}^i, \bar{e}_j, \bar{e}_k\right)$$
$$= {\phi^i}_\alpha {\phi_j}^\beta {\phi_k}^\gamma \underbrace{T(e^\alpha, e_\beta, e_\gamma)}_{{T^\alpha}_{\beta\gamma}}$$

Trace of mixed tensors

<u>Definition:</u>

indep. of pair of dual bases:

$$tr\, T = {T^i}_i$$

$$\bar{T^i}_i = \underbrace{{\phi^i}_\alpha {\phi_i}^\beta}_{{\delta_\alpha}^\beta} T\left(e^\alpha, e_\beta\right)$$
$$= {T^\alpha}_\alpha$$

In particular: $T = X \otimes \omega = X^i \omega_j e_i \otimes e^j$

$$\Rightarrow tr\, T = tr(X \otimes \omega) = X^i \omega_i = \langle \omega, X \rangle$$

Analogously:

$${T^i}_{jk} \overset{tr}{\longmapsto} S_k = {T^i}_{ik}$$

is linear map from type $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ to type $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

The tangent map

(or "differential map")

$$\varphi : M \longrightarrow \bar{M}$$

induces a linear map

$$\varphi_* : T_p(M) \longrightarrow T_{\bar{p}}(\bar{M})$$
$$X \longmapsto \varphi_* X$$

by either of the following definitions:

a) $(\varphi_* X)\,\bar{f} = X\left(\bar{f} \circ \varphi\right)$

b) let γ be a curve with tangent vector X at p Then $\varphi_* X$ is the tangent vector of $\bar{\gamma} = \varphi \circ \gamma$

Equivalence (b):

$$(\varphi_* X)\,\bar{f} = \frac{d}{dt}\underbrace{\bar{f}\left(\bar{\gamma}(t)\right)}_{\bar{f}\circ\varphi\circ\gamma}|_{t=0} = \frac{d}{dt}\left(\bar{f}\circ\varphi\right)(\gamma(t))\,|_{t=0} = X\left(\bar{f}\circ\varphi\right)$$

Components with respect to a basis $(e_1, \ldots, e_n)$ of T_p
$(\bar{e}_1, \ldots, \bar{e}_n)$ of $T_{\bar{p}}$

$$\bar{X} = \varphi_* X \text{ reads } X = X^k e_k$$
$$\bar{X}^i = \left\langle \bar{e}^i, X \right\rangle = X^k \underbrace{\left\langle \bar{e}^i, \varphi_* e_k \right\rangle}_{\equiv \varphi_*{}^i{}_k} = (\varphi_*)^i{}_k X^k$$

in particular with respect to canonical basis:

$$\bar{X}^i = \bar{X}(\bar{x}^i) = (\varphi_* X)(\bar{x}^i) = X(\bar{x}^i \circ \varphi) = X^k \frac{\partial \bar{x}^i}{\partial x^k}$$
$$\Rightarrow (\varphi_*)^i{}_k = \frac{\partial \bar{x}^i}{\partial x^k}$$

Adjoint φ^* of φ_*:

$$\varphi^* : T_{\bar{p}}{}^* \longrightarrow T_p{}^* \qquad \text{pull back}$$
$$\bar{\omega} \longmapsto \varphi^* \bar{\omega}$$
$$\text{by } \langle \varphi^* \bar{\omega}, X \rangle = \langle \bar{\omega}, \varphi_* X \rangle \qquad (X \in T_p)$$

or equivalent:

$$\varphi^* : d\bar{f} \longmapsto \varphi^*\left(d\bar{f}\right) = d\left(\bar{f}\circ\varphi\right) \qquad \left(\bar{f} \in \mathcal{F}_p\right)$$

In components:

$$\omega = \varphi^* \bar{\omega}$$

reads

$$\omega_k X^k = \bar{\omega}_i \left(\varphi_* X\right)^i = \bar{\omega}_i (\varphi_*)^i{}_k X^k$$

Mixed Tensors cannot be pushed forward/pulled back is general.
But: let φ be invertible in a neighbourhood of p with φ^{-1} smooth

$$\Longleftrightarrow \begin{cases} \dim\ M = \dim\ \bar{M} \\ \det\left(\frac{\partial \bar{x}^i}{\partial x^k}\right) \neq 0 \end{cases}$$

Then φ_*, φ^* are invertible and can be extended to tensors.

Definition: by example: $T, \bar{T}$ of type $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$(\varphi_* T)(\bar{\omega}, \bar{X}) = T(\varphi^* \bar{\omega}, {\varphi_*}^{-1} \bar{X})$$
$$(\varphi^* \bar{T})(\omega, X) = \bar{T}((\varphi^*)^{-1} \omega, \varphi_* X)$$
$$\Rightarrow \varphi_*, \varphi^* \text{ are } \underline{\text{inverse}} \text{ of one another}$$

Properties:

- $\varphi_*(T \otimes S) = (\varphi_* T) \otimes (\varphi_* S)$
- $tr(\varphi_* T) = \varphi_*(tr\ T)$

$$\text{e.g. } T = X \otimes \omega \rightarrow tr\ T = \langle \omega, X \rangle$$

$$tr(\varphi_* T) = tr(\varphi_*) \otimes \underbrace{(\varphi_* \omega)}_{(\varphi^*)^{-1}\omega}$$
$$= \left\langle (\varphi^*)^{-1} \omega \right\rangle = \left\langle \varphi^*((\varphi^*)^{-1} \omega), X \right\rangle$$
$$= <\omega, X> = \underbrace{tr\ T}_{\in \mathbb{R}} = \varphi_*(tr\ T)$$

Components: $\bar{T} = \varphi_* T$ reads

$$\bar{T}^i_k = \frac{\partial \bar{x}^i}{\partial x^\alpha} \frac{\partial x^\beta}{\partial \bar{x}^k} T^\alpha{}_\beta$$

Same expression for manifold-transformation as for coordinate-transformation!!!

1.2 Fields

Definition: A vector field on M is a linear map

$$X : \underbrace{\mathcal{F}}_{\substack{\text{smooth fct.} \\ \text{defined everywhere} \\ \text{on M}}} \longrightarrow \mathcal{F}$$

with product rule

$$X(fg) = (Xf)g + f(Xg)$$

Claim: $(Xf)(p)$ depends only on $\underbrace{f \in \mathcal{F}_p}_{\substack{\text{f in arbitrary small} \\ \text{neighbourhood of } p}}$

Proof: To show: $f = 0$ in neibourhood $U \ni p$, then $(Xf)(p) = 0$

Indeed: pick $g : M \to \mathbb{R}$, $\underbrace{\text{supp } g \subset U}_{\Rightarrow fg=0}$, $g(p) = 1$

$$\begin{aligned} 0 &= X(fg)(p) \\ &= (Xf)(p)\underbrace{g(p)}_{1} + \underbrace{f(p)}_{0}(Xg)(p) \\ &\Rightarrow (Xf)(p) = 0 \end{aligned}$$

Hence: for any $p \in M$:

$$X_p : \underbrace{f}_{\in \mathcal{F}} \mapsto (Xf)(p)$$

defines $X_p \in T_p$

In any chart: $X = X^i(x)\frac{\partial}{\partial x^i}$ with $X^i = Xx^i$

Thus: a vector field can also be viewed as

- an assignment $p \mapsto X_p$ with smooth coordinates (in any chart)
- a linear differential operator of 1^{st} order

Operators on vector fields:

$$\begin{aligned} X &\longmapsto fX && \text{(multiplication by } f \in \mathcal{F}) \\ X, Y &\longmapsto [X, Y] = XY - YX && \text{(commutator, Lie-Bracket)} \end{aligned}$$

$[X, Y]$ enjoys product rule, unlike XY:

$$\begin{aligned} (XY)(fg) &= X\left((Yf)\,g + f\,(Yg)\right) \\ &= (XYf)\,g + (Yf)\,(Xg) + (Xf)\,(Yg) + f\,(XYg) + f\,[X, Y]\,g \end{aligned}$$

$$[X, Y]\,(fg) = ([X, Y]\,f)\,g$$

Jacobi-Identity:

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$$

Covector Fields: **(or 1-forms)**

$$\begin{aligned} \omega : \text{ vector fields } (M) &\longrightarrow \mathcal{F} \\ X &\longmapsto \omega(X) \end{aligned}$$

with

$$\omega(fX) = f\omega(X) \qquad (f \in \mathcal{F}) \qquad (\text{"f-linearity})$$

<u>Fact:</u> $\omega(X)(p)$ depends only on $X_p \in T_p$

Hence: for any $p \in M$

$$\omega(X)(p) = \langle \omega_p, X_p \rangle$$

defines a covector $\omega_p \in {T_p}^*$

In any chart:

$$\omega = \omega_i(x) dx^i \qquad \text{with } \omega_i = \left\langle \omega, \frac{\partial}{\partial x^i} \right\rangle \in \mathcal{F}$$

smooth components

<u>Caution:</u> not every ω is of the form $\omega = df$

(otherwise $\omega_i = \frac{\partial f}{\partial x^i} \to \frac{\partial \omega_i}{\partial x^j} = \frac{\partial^2 f}{\partial x^i \partial x^j} = \frac{\partial \omega_j}{\partial x^i}$
which is false in general!)

Tensorfields

<u>Definition:</u> Tensor field of type $\otimes_2^1$ is a function

$$R : \underbrace{\omega}_{1-\text{form}}, \underbrace{X, Y}_{\substack{\text{vector}\\ \text{fields}}} \longmapsto R(\omega, X, Y) \qquad \in \mathcal{F} \qquad \text{f-linear in all arguments}$$

Equivalently:

$$R : p \in M \longrightarrow R_p \text{ tensor on } T_p$$

with smooth coordinates.

Tangent map:

1-forms:

$$\bar{\omega} \longmapsto \varphi^* \bar{\omega} \qquad \text{pointwise}$$

$$(\varphi^* \bar{\omega})_p = \varphi^* \bar{\omega}_{\varphi(p)} \qquad \varphi^* : {T_{\bar{p}}}^* \to {T_p}^*$$

Let $\varphi : M \to \bar{M}$ be a <u>global</u> diffeomorphism (φ^{-1} exists, smooth)
vectorfields:

$$X \longmapsto \varphi_* X$$

$$(\varphi_* X)_{\bar{p}} = \varphi_* X_{\varphi^{-1}(\bar{p})}$$

equivalently:

$$(\varphi_* X)\,\bar f = \left[X\left(\bar f \circ \varphi\right)\right] \circ \varphi^{-1}$$

Note:

$$\varphi_* [X,Y] = [\varphi_* X, \varphi_* Y]$$

Flows and generating fields

Definition: A flow on M is

- 1-parameter group of diffeomorphisms

$$\varphi_t : M \longrightarrow M, \qquad (t \in \mathbb{R})$$

with

$$\varphi_t \circ \varphi_s = \varphi_{t+s}$$

(in particular $\varphi_0 \circ \varphi_0 = \varphi_0 \Rightarrow \varphi_0$ identity on M $\Rightarrow {\varphi_t}^{-1} = \varphi_{-t}$)

- Orbit (or integral curve) of $p \in M$

$$t \longmapsto \varphi_t(p) = \gamma(t)$$

is smooth in t

A flow φ_t determines a vector field X (the generating field)

$$Xf = \frac{d}{dt}(f \circ \varphi_t)\Big|_{t=0}$$

i.e. $X_p = \frac{d\varphi_t(p)}{dt}\Big|_{t=0} = \dot\gamma(0)$ (tangent vector to the orbit of p at p)

At any point $\gamma(t)$

$$\dot\gamma(t) = \frac{d\varphi_t(p)}{dt} = \frac{d}{ds}\varphi_{t+s}(p)\Big|_{s=0} = \frac{d}{ds}\varphi_s(\varphi_t(p))\Big|_{s=0} = X_{\varphi_t(p)}$$

Hence: $\gamma(t)$ sets ODE

$$\dot\gamma(t) = X_{\gamma(t)}, \qquad \gamma(0) = p$$

$\Rightarrow$ generating vector field determines $\varphi_t(p) = \gamma(t)$

1.3 The Lie derivative

Directional derivative of a function f in direction X:

$$XF = \frac{d}{dt} f(\varphi_t(p))\Big|_{t=0} = \lim_{t\to\infty} \frac{f(\varphi_t(p)) - f(p)}{t}, \qquad Xf = X^k f_{,k}$$

Derivation of a vector field?

componentwise $X^k Y^i_{,k}$? **NO!!!**

$\Rightarrow Y_p, Y_{\varphi_t(p)}$ elements of different tangent spaces!

$\Longrightarrow$ Before diff. can be taken, $Y_{\varphi_t(p)}$ has to be transported to T_p.
One possibility: by means of tangent map φ_{t*} (Lie-Transport)

Definition: The Lie derivative $L_X R$ of a tensor field R in direction of the vector field X is

$$L_X R = \frac{d}{dt}\left(\varphi_t{}^* R\right)\Big|_{t=0}$$

$$\text{i.e.} \qquad (L_X R)_p = \frac{d}{dt}\varphi_t{}^* R_{\varphi_t(p)}\Big|_{t=0}$$

$\Longrightarrow \varphi_t{}^* R$ is a tensor over $p \quad \forall\, t$

Coordinate expressions: In any chart around $p \in M$, for small t,

$$\varphi_t : \longmapsto \bar{x}(t,x) \qquad M = \bar{M} \Rightarrow x, \bar{x} \in \text{ same chart}$$

$$\text{satisfies } \frac{\partial \bar{x}^i}{\partial t} = X^i\left(\bar{x}(t)\right)$$

Taylor-Expansion:

$$\bar{x}^i(t,x) = x^i + tX^i(x) + \mathcal{O}\left(t^2\right)$$
$$x^i(t,x) = \bar{x}^i + tX^i\left(\bar{x}\right) + \mathcal{O}(t^2)$$

Hence:

$$\frac{\partial \bar{x}^i}{\partial x^k} = \delta^i{}_k + tX^i{}_{,k} + \mathcal{O}(t^2)$$
$$\frac{\partial^2 \bar{x}^i}{\partial t \partial x_k} = X^i{}_{,k}$$
$$\frac{\partial x^i}{\partial \bar{x}^k} = \delta^i{}_k - tX^i{}_{,k}(\bar{x}) + \mathcal{O}(t^2)$$
$$\frac{\partial x^i}{\partial \bar{x}^k}\Big|_{\bar{x}=\bar{x}(t,x)} = \delta^i{}_k - tX^i{}_{,k}(x) + \mathcal{O}(t^2)$$
$$\frac{\partial^2 x^i}{\partial t \partial \bar{x}_k} = -X^i{}_{,k}$$

$$\left(\varphi_t{}^* R_{\varphi_t(p)}\right)^i{}_j(x) = R^\alpha{}_\beta(\bar{x})\frac{\partial x^i}{\partial \bar{x}^\alpha}\Big|_{\bar{x}=\bar{x}(t,x)}\frac{\partial \bar{x}^\beta}{\partial x^j} \qquad R \otimes^1_1$$

derivative at $t = 0$:

$$(L_X R)^i{}_j(x) = R^{\not{\alpha}\, i}{}_{\not{\beta}\, j,k} X^k - \underbrace{R^\alpha{}_j X^i{}_{,\alpha} + R^i{}_\beta X^\beta{}_j}_{\text{contribution of the Lie-Transport}}$$

$\downarrow$ component wise

Properties of L_X:

a) L_X is a linear map from tensor fields to tensor fields of the same type
b) $L_X(tr\,T) = tr(L_X T)$ any trace
c) $L_X(T \otimes S) = (L_X T) \otimes S + T \otimes (L_X S)$
d) $L_X f = Xf \qquad f \in \mathcal{F}(M)$
e) $L_X Y = [X,Y] = XY - YX$ (Y vector field on M)

Proof: (a):$\surd$, (b):(pullback analogy)$\surd$
(c): ${\varphi_t}^*(T \otimes S) = ({\varphi_t}^* T) \otimes ({\varphi_t}^* S)$
(d): $L_X f = \frac{d}{dt} {\varphi_t}^* f \Big|_{t=0} = \frac{d}{dt}(f \circ \varphi_t)\Big|_{t=0} = Xf$

$$\begin{aligned}
\text{(e): } (L_X Y)f &= \left(\tfrac{d}{dt}{\varphi_t}^* Y\,|_{t=0}\right) f = \tfrac{d}{dt}\left(\varphi_{-t*}Y\right) f\,|_{t=0} = \tfrac{d}{dt}\left(Y(f\circ\varphi_{-t})\circ\varphi_t\right)|_{t=0} \\
&= Y\left(\tfrac{d}{dt}(f\circ\varphi_{-t})\right)|_{t=0} + \tfrac{d}{dt}(Yf)\circ\varphi_t\,|_{t=0} \\
&= Y(-Xf) + XYf = [X,Y]\,f
\end{aligned}$$

<u>Alternative</u> definition of L_X (not making use of flows):

Claim: for given X, the map L_X is uniquely determined by (a-e)
(hence agrees with the previous definition)

Proof: (d): L_X is uniquely determined on tensor fields of type $\otimes^0_0$
(e): " " $\otimes^1_0$
Will show: " " $\otimes^0_1$
(c): " " $\otimes^p_q$
ω : 1-form, Y vector field

$$\omega(Y) = tr\,(Y \otimes \omega)$$

$$\begin{aligned}
(L_X\omega)(Y) = tr\,(Y\otimes L_X\omega) &= tr\,(L_X(Y\otimes\omega)) - tr(\underbrace{(L_X Y)}_{[X,Y]}\otimes\omega) \\
&= L_X\,\underbrace{tr(Y\otimes\omega)}_{\omega(Y)} - \omega([X,Y]) \\
&= X(\omega(Y)) - \omega([X,Y])
\end{aligned}$$

<u>Further Properties:</u>

- L_X is linear in X, but not f-linear: $L_{\lambda X} = \lambda L_X \qquad \lambda \in \mathbb{R} \qquad L_{fX} \neq f L_X$

$$\begin{aligned}
L_{fX}Y = [fX, Y] &= fXY - YfX = fXY - (Yf)X - fYX \\
&= f[X,Y] - (Yf)X \\
&= (fL_X)Y - (Yf)X \neq (fL_X)Y
\end{aligned}$$

- $L_{[X,Y]} = L_X L_Y - L_Y L_X$

<u>Meaning of $[X,Y]=0$</u>

$$\varphi_t \leftrightarrow X : (Xf)(x) = \frac{d}{dt}(f\circ\varphi_t)\Big|_{t=0}$$

$$\psi_s \leftrightarrow Y : (Yg)(x) = \frac{d}{ds}(g\circ\psi_s)\Big|_{s=0}$$

Theorem: $[X,Y]=0 \Leftrightarrow \varphi_t \circ \psi_s = \psi_s \circ \varphi_t$

Proof: " $\Leftarrow$ " $(f\circ\varphi_t)\circ\psi_s = (f\circ\psi_s)\circ\varphi_t$

$\frac{d}{dt}\cdots\Big|_{t=0}$ $Xf\circ\psi_s = X(f\circ\psi_s)$

$\frac{d}{ds}\cdots\Big|_{s=0}$ $YXf = XYf$

2 Affine Connections

2.1 Parallel Transport and covariant derivaties

Definition: Any curve γ in M is equipped with a parallel transport

$$\tau(t,s): T_{\gamma(s)} \longrightarrow T_{\gamma(t)}$$
$$X(s) \longmapsto X(t)$$
$$\searrow$$
$$\text{chart:} X^i(t) = \tau^i{}_k(t,s)X^k(s)$$

Satisfying

- linear with $\tau(t,s)\tau(t,r)$ $\quad \tau(t,t)=1$
- in any chart:

$$\frac{\partial}{\partial t}\tau^i{}_k(t,s)\Big|_{t=s} = -\overbrace{\Gamma^i{}_{lk}\left(\gamma(s)\right)}^{\text{Christoffel-Symbols of transport } \tau}\dot\gamma^l(s) \circledast$$

$\searrow$ convention

$\rightarrow$ linear in $\dot\gamma^l$

Remarks:

1. Lie-Transport φ_{t*} along an orbit of the vector field has

$$(\varphi_{t*})^i{}_j = \delta^i{}_j + tY^i{}_{,j} + \mathcal{O}(t^2)$$
$$\frac{\partial}{\partial t}(\varphi_{t*})^i{}_j\big|_{t=0} = Y^i{}_{,j}$$

does not depend on $\dot\gamma^l(0) = Y^l(x)$ only
Hence not of the form $\circledast$

2. Parallel transported vector

$$X(t) = \tau(t,s)X(s)$$

satisfies (in any chart) the ODE

$$\begin{aligned}\dot X^i(t) &= \frac{\partial}{\partial t}\left(\tau(t,s)X(s)\right)^i = \frac{\partial}{\partial\lambda}\left(\tau(t+\lambda,s)X(s)\right)^i\big|_{\lambda=0}\\ &= \frac{\partial}{\partial\lambda}\tau^i{}_k(t+\lambda,s)\big|_{\lambda=0}\left(\tau(t,s)X\right)^k\\ &= -\Gamma^i{}_{lk}\left(\gamma(t)\right)\dot\gamma^l(t)X^k(t)\end{aligned}$$

i.e.

$$\dot{X}^i(t) + \Gamma^i{}_{lk}\left(\gamma(t)\right)\dot{\gamma}^l(t)X^k(t) = 0$$

Note: the $\dot{X}^i$ are not the components of a vector, nor are the $\Gamma^i{}_{lk}$ those of a tensor field

3. Linearity of ⊛ with respect to $\dot{\gamma}^l$ implies:

   $\tau(t,s)$ is independent of parametrization of γ (but does depend on γ, i.e. not just on endpoints $\gamma(t), \gamma(s)$)
   More precisely: reparametrization $r : \tilde{t} \longrightarrow t$ (monotonic)

   $$\tilde{\gamma}\left(\tilde{t}\right) = \gamma(t)\big|_{t=r\left(\tilde{t}\right)}$$

   Claim: $\tilde{\tau}\left(\tilde{t},\tilde{s}\right) = \tau(t,s)$ i.e.
   if $\tilde{X}\left(\tilde{s}\right) = X(s)$ and $X(t) = \tau(t,s)X(s)$, $\tilde{X}\left(\tilde{t}\right) = \tilde{\tau}\left(\tilde{t},\tilde{s}\right)\tilde{X}\left(\tilde{s}\right)$
   then $\tilde{X}\left(\tilde{t}\right) = X(t)$

   $$\underbrace{\frac{d\tilde{X}^i}{d\tilde{t}}}_{=\frac{d\tilde{X}^i}{dt}\frac{dt}{d\tilde{t}}} = -\Gamma^i{}_{lk}\left(\tilde{\gamma}\left(\tilde{t}\right)\right)\underbrace{\frac{d\tilde{\gamma}^l}{d\tilde{t}}}_{\frac{d\tilde{\gamma}^l}{dt}\frac{dt}{d\tilde{t}}}\tilde{X}^k\left(\tilde{t}\right) \text{ is ODE satisfied by } X^i(t)$$

   $\Rightarrow$ same starting point $X(s)$, same ODE
   $\Rightarrow$ solutions of this ODE are the same

4.
$$\tau(t,s)\tau(s,t) = \tau(t,t) = 1$$
$$\tau^i{}_k(t,s)\tau^k{}_i(s,t) = \delta^i{}_k$$
$$\frac{\partial}{\partial s}\dots\big|_{s=t} : \quad \frac{\partial}{\partial s}\tau^i{}_k(t,s)\big|_{s=t}\delta^k{}_j - \delta^i{}_k\Gamma^k{}_{lj}\left(\gamma(t)\right)\dot{\gamma}^l(t) = 0$$
$$\Rightarrow \frac{\partial}{\partial s}\tau^i{}_j(t,s)\big|_{s=t} = \Gamma^i{}_{lj}\left(\gamma(t)\right)\dot{\gamma}^l(t)$$

5. Change of chart $x \leftrightarrow \bar{x}$:

$$\bar{\tau}^i_k(t,s) = \tau^p{}_q(t,s)\frac{\partial \bar{x}^i}{\partial x^p}\Big|_{\gamma(t)}\frac{\partial x^q}{\partial \bar{x}^k}\Big|_{\bar{\gamma}(s)}$$

$$\frac{\partial}{\partial s}\bar{\tau}^i_k(t,s)\big|_{s=t} = \Gamma^p{}_{rq}\underbrace{\dot{\gamma}^r}_{\frac{\partial x^r}{\partial \bar{x}^l}\dot{\bar{\gamma}}^l}\frac{\partial \bar{x}^i}{\partial x^p}\frac{\partial x^q}{\partial \bar{x}^k} + \delta^p{}_q\frac{\partial \bar{x}^i}{\partial x^p}\frac{\partial^2 x^q}{\partial \bar{x}^k\partial \bar{x}^l}\dot{\bar{\gamma}}^l$$
$$= \bar{\Gamma}^i_{lk}\dot{\bar{\gamma}}^l$$

Hence:

$$\bar{\Gamma}^i_{kl} = \Gamma^p{}_{rq}\frac{\partial \bar{x}^i}{\partial x^p}\frac{\partial x^r}{\partial \bar{x}^l}\frac{\partial x^q}{\partial \bar{x}^k} \qquad ⊛⊛$$

Conversely: any arbitrary $\Gamma^p{}_{rq}(x)$ transforming as ⊛⊛ upon change of coordinates defines a parallel transport.

Parallel Transport of Tensors:

should satisfy:

$$\tau(t,s)\,(T\otimes S) = \tau(t,s)T\otimes\tau(t,s)S$$
$$\tau(t,s)\,(tr\,T) = tr\,\tau(t,s)T \qquad \text{any trace}$$
$$\tau(t,s)c = c \qquad c\in\Re$$

This extends the transport from vectors to tensors in a unique way:
Hence

- for a covector $\omega\in {T_{\gamma(s)}}^*$ recall $tr\,(\omega\otimes X) = \langle\omega, X\rangle$ apply $\tau(t,s)$:

$$<\tau(t,s)\omega, \underbrace{\tau(t,s)X}_{=:\tilde{X}}>_{\gamma(t)} = \langle\omega, X\rangle_{\gamma(s)}$$
$$\left\langle\tau(t,s)\omega,\tilde{X}\right\rangle = \left\langle\omega,\tau(s,t)\tilde{X}\right\rangle$$

Compute:

$$\underbrace{(\tau(t,s)\omega)_k}_{\equiv\,{\tau_k}^i(t,s)\omega_i}\tilde{X}^k = \omega_i{\tau^i}_k(s,t)\tilde{X}^k \to {\tau_k}^i(t,s) = {\tau^i}_k(s,t)$$

- for a tensor of type $\otimes^1_1$:

$$(\tau(t,s)T)^i{}_k = {\tau^i}_\alpha{\tau_k}^\beta{T^\alpha}_\beta$$

Covariant Derivative corresponding to τ

X vector field, R tensor field

$$(\nabla_X R)_p = \frac{d}{dt}\tau(0,t)R_{\gamma(t)}\Big|_{t=0}$$

for any curve $\gamma(t)$ with $\gamma(0)=p$ and $\dot{\gamma}(0) = X_p$

Properties:

a) maps tensor field to tensor field of same type
b) $\nabla_X f = Xf \qquad \Big((\nabla_X f)_p = \frac{d}{dt}\tau(0,t)f(\gamma(t))\big|_{t=0} = \dot{\gamma}(0)f = X_p f = (Xf)_p\Big)$
c) $\nabla_X\,(tr\,T) = tr\,\,(\nabla_X T)$
d) $\nabla_X\,(T\otimes S) = (\nabla_X T)\otimes S + T\otimes(\nabla_X S)$

Definition: If the covariant derivative ∇_X acts on vector fields Y, we call it an **affine connection $\nabla_X Y$.**

Properties of $\nabla_X Y$:

(i) $\nabla_X Y$ is a vector field, linear in X,Y
(ii) $\nabla_X Y$ is f-linear: $\nabla_{fX}Y = f\nabla_X Y$ (unlike L_X)
(iii) $\nabla_X(fY) = f\nabla_X Y + XfY$

Proof: (ii) in any chart:

$$\gamma^i(t)= x^i + tX^i(x) + \mathcal{O}(t^2)$$

$$(\nabla_X Y)^i = \frac{d}{dt}\tau^i{}_k(0,t)Y^k\left(x^1 + tX^1(x) + \cdots + x^n + tX^n(x)\right)\Big|_{t=0}$$
$$= \delta^i{}_k Y^k{}_{,l}X^l + \Gamma^i{}_{lk}X^lY^k$$

i.e.

$$(\nabla_X Y)^i = \left(Y^i{}_{,l} + \Gamma^i{}_{lk}Y^k\right)X^l \circledast\circledast\circledast$$

$(\nabla_X Y)_p$ depends only on X at $p \hat{=} x$

(iii) $$(\nabla_X(fY))_p = \frac{d}{dt}\tau(0,t)\underbrace{(fY)_{\gamma(t)}}_{f(\gamma(t))Y_{\gamma(t)}}\Big|_{t=0} = \frac{d}{dt}f(\gamma(t))\,\tau(0,t)Y_{\gamma(t)}\Big|_{t=0}$$
$$= f(p)\,(\nabla_X Y)_p + (Xf)_pY_p$$

Conversely: Any action $\nabla_X Y$ (i.e. satisfying (i-iv)) defines a parallel transport (bijective) with respect to the canonical basis:

$$\nabla_X Y = \nabla_X\left(Y^ie_i\right)= \left(XY^i\right)e_i + Y^k\nabla_X e_k$$
$$= X^lY^i_{,l}e_i + Y^kX^l\nabla_{e_l}e_k$$
$$(\nabla_X Y)^i= X^lY^i_{,l} + Y^kX^l\left\langle e^i,\nabla_{e_l}e_k\right\rangle$$
$$= \left(Y_{,l} + \left\langle e^i,\nabla_{e_l}e_k\right\rangle Y^k\right)X^l$$

$\Rightarrow$ same as $\circledast\circledast\circledast$ with $\Gamma^i{}_{lk} = \left\langle e^i,\nabla_{e_l}e_k\right\rangle$

Bijective relation $\tau \leftrightarrow \nabla$

Let $\dot\gamma(t) = X_{\gamma(t)}$ Then $Y(s) = Y(\gamma(s))$

$$Y(t) = \tau(t,s)Y(s) \quad\Leftrightarrow\quad \nabla_X Y = 0$$

note $(\nabla_X Y)^i = \left(Y^i{}_{,l} + \Gamma^i{}_{lk}Y^k\right)\dot\gamma^l = \dot Y^i + \Gamma^i{}_{lk}Y^k\dot\gamma^l$

The covariant derivative ∇: (not ∇_X)

Definition: by example: T of type $\otimes^1_1$: i.e. $T(\omega,Y)$ f-linear in ω : 1-form
Y : vector field

then $(\nabla_X T)(\omega,Y) =: (\nabla T)(\omega,Y,X)$ defines a tensor field of type $\otimes^1_2$

Components:

$$(\nabla T)^i{}_{kl} = T^i{}_{k;l}$$

with respect to canonical basis:

-for vector field Y:

$$Y^i{}_{;k} = (\nabla Y)^i{}_k = (\nabla Y)(e^i,e_k) = (\nabla_{e_k}Y)(e^i)$$
$$= (\nabla_{e_k}Y)^i = Y^i{}_{,k} + \Gamma^i{}_{kl}Y^l$$

-for 1-form ω:

$$\omega_{i;k} = (\nabla\omega)_{ik} = (\nabla\omega)(e_i,e_k) = (\nabla_{e_k}\omega)(e_i)$$
$$= e_k\underbrace{\omega(e_i)}_{\omega_i} - \omega\left(\nabla_{e_k}e_i\right)$$
$$= \omega_{i,k} - \Gamma^l{}_{ki}\omega_l$$

-Tensor field of type $\otimes^1_1$:

$$T^i{}_{j;k}= T^i{}_{j,k} + \Gamma^i{}_{kl}T^l{}_j - \Gamma^l{}_{kj}T^i{}_l$$

Remark:

- $T^i{}_{j,k}$ depends only on $T^i{}_j(x)$ for given i, j (in a neighbourhood of x)
- $T^i{}_{j;k}$ depends on all components $T^\alpha{}_\beta(x)$

2.2 Torsion and Curvature

Let ∇ be an affine connection on M, X, Y, Z vector fields
Definition:

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]$$
$$R(X,Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}$$

- At first:
  - $T(X,Y)$ is a vector field
  - $R(X,Y)$ is a linear map from tensor fields to tensor fields of the same type
- $\left.\begin{array}{l} T(X,Y) = -T(Y,X) \\ R(X,Y) = -R(Y,X) \end{array}\right\}$ antisymmetry
- $T(X,Y)$ is f-linear in X, Y

  Hence
  $$(\omega, X, Y) \longmapsto \langle \omega, T(X,Y) \rangle =: T^i{}_{jk} \omega_i X^j Y^k$$
  is a tensor field of type $\otimes^1_2$

  proof: (of f-linearity)
  $$[fX, Y] = (fX)Y - \underbrace{Y(fX)}_{(Yf)X + f(YX)} = f\,[X,Y] - (Yf)X$$
  hence
  $$T(fX,Y) = f\nabla_X Y - f\nabla_Y X - (Yf)X - F\,[X,Y] + (Yf)X = fT(X,Y)$$
- $R(X,Y)Z$ (a vector field) is f-linear in X, Y, Z

  Hence
  $$(\omega, Z, X, Y) \longmapsto \langle \omega, R(X,Y)Z \rangle =: R^i{}_{jkl} \omega_i Z^j X^k Y^l$$
  defines a tensor field R of type $\otimes^1_3$

  (curvature or Riemann tensor)

  proof: (of f-linearity)
  $$\begin{aligned} R(fX,Y) &= f\nabla_X\nabla_Y - \nabla_Y f \nabla_X - f\nabla_{[X,Y]} + (Yf)\nabla_X \\ &= f\nabla_X\nabla_Y - f\nabla_Y\nabla_X - (Yf)\nabla_X - f\nabla_{[X,Y]} + (Yf)\nabla_X \\ &= fR(X,Y) \end{aligned}$$
  f-linearity in Z: see next proposition, part (d).

Proposition:

a) $R(X,Y)f = 0$

b) $R(X,Y)(S \otimes T) = (R(X,Y)S) \otimes T + S \otimes (R(X,Y)T)$

c) $tr\ R(X,Y)T = R(X,Y)tr\ T$ (any trace not contracting X or Y)

d) $\langle \omega, R(X,Y)Z \rangle = -\langle R(X,Y)\omega, Z \rangle$

Proof:

a) $R(X,Y)f = XYf - YXf - [X,Y]f = 0$

b)

$$
\begin{aligned}
R(X,Y)(S \otimes T) &= \nabla_X \underbrace{((\nabla_Y S) \otimes T + S \otimes (\nabla_Y T)) - \nabla_Y ((\nabla_X S) \otimes T + S \otimes (\nabla_X T))}_{\text{mixed terms drop out}} \\
&- \left(\nabla_{[X,Y]} S\right) \otimes T - S \otimes \left(\nabla_{[X,Y]} T\right) \\
&= \cdots = (R(X,Y)S) \otimes T + S \otimes (R(X,Y)T)
\end{aligned}
$$

c) follows from $\nabla_X tr\ T = tr\ \nabla_X T$

d) From a)-c) we have

$$
\begin{aligned}
0 = R(X,Y)\langle \omega, Z \rangle &= R(X,Y) tr\ (\omega \otimes Z) = tr\ R(X,Y)(\omega \otimes Z) \\
&= tr\ ((R(X,Y)\omega) \otimes Z) + tr\ (\omega \otimes R(X,Y)Z) \\
&= \langle R(X,Y)\omega, Z \rangle + \langle \omega, R(X,Y)Z \rangle
\end{aligned}
$$

Components with respect to coordinate basis: $e_i = \frac{\partial}{\partial x^i}, e^i = dx^i$

$\Rightarrow [e_i, e_j] = 0$

- $T^k{}_{ij} = T(e^k, e_i, e_j) = \left\langle e^k, T(e_i, e_j) \right\rangle = \left\langle e^k, \nabla_{e_i} e_j - \nabla_{e_j} e_i \right\rangle = \Gamma^k{}_{ij} - \Gamma^k{}_{ji}$
  In particular:
  $$T = 0 \Longleftrightarrow \Gamma^k{}_{ij} = \Gamma^k{}_{ji}$$

- $$
\begin{aligned}
R^i{}_{jkl} &= \left\langle e^i, R(e_k, e_l) e_j \right\rangle = \left\langle e^i, \left(\nabla_{e_k} \nabla_{e_l} - \nabla_{e_l} \nabla_{e_k}\right) e_j \right\rangle \\
&= \left\langle e^i, \nabla_{e_k} \left(\Gamma^s{}_{lj} e_s\right) - \nabla_{e_l} \left(\Gamma^s{}_{kl} e_s\right) \right\rangle \\
&= \Gamma^s{}_{lj,k} \left\langle e^i, e_s \right\rangle + \Gamma^s{}_{lj} \Gamma^i{}_{ks} - \Gamma^s{}_{kj,l} \left\langle e^i, e_s \right\rangle - \Gamma^s{}_{kj} \Gamma^i{}_{ls} \\
&= \Gamma^i{}_{lj,k} - \Gamma^i{}_{kj,l} + \Gamma^s{}_{lj} \Gamma^i{}_{ks} - \Gamma^s{}_{kj} \Gamma^i{}_{ls}
\end{aligned}
$$

Bianchi Identities **(in the special case of torsion= 0)**

1. $R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0$
2. $(\nabla_X R)(Y,Z) + (\nabla_Y R)(Z,X) + (\nabla_Z R)(X,Y) = 0$

   Proof: Let $X_1 = X, X_2 = Y, X_3 = Z$. Then

   $$\sum_{i=1}^{3} R(X_i, X_{i+1})X_{i+2} = \sum_{i=1}^{3} (\nabla_{X_i} \underset{i\to i+2}{\underset{\downarrow}{\nabla_{X_{i+1}}}} X_{i+2} - \nabla_{X_{i+1}} \underset{i\to i+1}{\underset{\downarrow}{\nabla_{X_i}}} X_{i+2} - \nabla_{[X_i,X_{i+1}]} X_{i+2})$$

   seperate in 3 sums and replace

   $$= \sum_{i=1}^{3} (\underbrace{\nabla_{X_{i+2}} \nabla_{X_i} X_{i+1} - \nabla_{X_{i+2}} \nabla_{X_{i+1}} X_i}_{\substack{\nabla_{X_{i+2}}[X_i,X_{i+1}] \\ \text{since } \nabla_{X_i}X_{i+1} - \nabla_{X_{i+1}}X_i \\ = \underbrace{T(X_i,X_{i+1})}_{=0, \text{ by assumption}} + [X_i,X_{i+1}]}} - \nabla_{[X_i,X_{i+1}]} X_{i+2})$$

   $$= \sum_{i=1}^{3} T(X_{i+2}, [X_i, X_{i+1}]) + \sum_{i=1}^{3} [X_{i+2}, [X_i, X_{i+1}]] = 0$$

   by the Jacobi-Identity.

Geometric meaning of the curvature

Let X, Y be vector fields, φ_t, ψ_s the corresponding flows, and assume $[X,Y] = 0$

$$\Longleftrightarrow \begin{cases} R(X,Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X \\ \varphi_t \circ \psi_s = \psi_s \circ \varphi_t \end{cases}$$

Consider

$\tau_X(t) : T_P \longrightarrow T_{\varphi_t(p)}$
parallel transport along the orbit $\varphi_t(p)$ of X

$\tau_Y(s) : T_P \longrightarrow T_{\psi_s(p)}$
parallel transport along the orbit $\psi_s(p)$ of Y

Transport Z around the loop:

$$Z(t,s) = \tau_Y(-s)\tau_X(-t)\tau_Y(s)\tau_X(t)Z$$

Expand this in Taylor-Series with respect to t and s:

$$Z(0,0) = Z$$
$$Z(t,0) = Z \qquad Z(0,s) = Z$$
$$Z(t,s) = Z + \left.\frac{\partial^2 Z}{\partial t \partial s}\right|_{t=s=0} ts + \text{higher order}$$

Remember that we have $\frac{d}{dt}\tau_X(t)Z \mid_{t=0} = -\nabla_X Z$

hence

$$\left.\frac{\partial Z}{\partial t}\right|_{t=0} = \tau_Y(-s)\nabla_X \tau_Y(s) Z - \nabla_X(Z)$$

$$\left.\frac{\partial^2 Z}{\partial t \partial s}\right|_{t=s=0} = \left(\nabla_Y \nabla_X - \nabla_X \nabla_Y\right) Z = -R(X,Y)Z$$

Thus:

$$Z(t,s) = Z - ts \cdot R(X,Y)Z + \text{higher order}$$

Curvature measures the difference of a vector before and after having circulated around the loop.

3 Pseudo-Riemannian manifolds

Let M be equipped with a pseudo-Riemannian metric: a symmetric, non-degenerate tensor field of type $\otimes^0_2$,

$$g(X,Y) \equiv (X,Y)$$

non-degenerate: $\forall p \in M:$ If $X \in T_p,\ g_p(X,Y) = 0\, \forall Y \in T_p$

$$\implies X_p = 0$$

in components:

$$g(X,Y) = g_{ik} X^i Y^k$$

(non-degenerate $\Leftrightarrow \det(g) \neq 0$)

$$g_{ik} = g_{ki} \quad \det(g_{ik}) \neq 0$$

Remark:
Riemannian metric: instead of non-degeneracy only requires the stronger: $g_p(X,X) \geq 0\, \forall X \in T_p,\ g_p(X,X) = 0 \Leftrightarrow X = 0$ (not assumed here)
By means of a metric, identify vector fields and 1-forms:

$$g : \underset{\substack{\uparrow \\ \text{vector field}}}{X} \mapsto \underset{\substack{\uparrow \\ \text{1-form}}}{gX} = \tilde{X} \qquad\qquad \underset{\substack{\uparrow \\ \text{1-form}}}{\omega} \mapsto \underset{\substack{\uparrow \\ \text{vector field}}}{g^{-1}\omega} = \tilde{\omega}$$

through

$$\langle gX, Y\rangle := g(X,Y) \qquad\qquad \langle \omega, Y\rangle = g\left(g^{-1}\omega, Y\right)$$

In components:

$$\tilde{X}_i Y^i = g_{ik} X^k Y^i \quad \rightsquigarrow \quad \tilde{X}_i = g_{ik} X^k \quad \text{"lowering the index"}$$

$$\omega_i Y^i = g_{ik} \tilde{\omega} Y^i \qquad\qquad \tilde{\omega}^i = g^{ik}\omega_k \quad \text{"raising the index"}$$

Remark: Let $(e_1, \dots, e_n)$, $(e^1, \dots, e^n)$ be dual basis of T_p, $T_p{}^*$. Then

$$\tilde{e}^j = g^{-1} e^j \in T_p$$

$$\left.\begin{array}{l}(\tilde{e}^j, X) = \langle e^j, X\rangle = X^j \\ (g_{ij}\tilde{e}^j, X) = g_{ij} X^j = (e_i, X)\end{array}\right\} e_i = g_{ij}\tilde{e}^j$$

Note: If $g_{ij} = \delta_{ij}$, then $e_i = \tilde{e}^i$; only if g is positive definite.

From now on: drop the $\sim$:

$$\left.\begin{array}{l}X^i \text{ contravariant} \\ X_i \text{ covariant}\end{array}\right\} \text{components of the same vector}$$

Similary: identify tensors of type$\otimes_q^p$, $\otimes_{q'}^{p'}$ for $p + q = p' + q'$:

for example:

$$T^i{}_k = g^{il} T_{lk} = g_{kl} T^{il}$$

(consistency of $\left(g^{ik}\right)$: raise indices of g_{ik}:

$$g^{ji} g^{lk} g_{ik} = g^{lk} \delta^j{}_k = g^{lj}$$

hence is consistent)

A particular connection is distinguished by the metric:

Theorem: (Riemann or Levi-Civita connection)

There is precisely one affine connection ∇ with:

1. Torsion $T = 0$
2. $\nabla g = 0$

Theorem is only as good as its assumptions, they have to be justified physically, ultimately by from the equivalence principle

("∇ is symmetric and metric")
In fact it is given as

$$2\left(\nabla_X Y, Z\right) = X(Y, Z) + Y(Z, X) - Z(X, Y) - (X, [Y, Z]) + (Y, [Z, X]) + (Z, [X, Y]) \circledast\circledast\circledast\circledast$$

proof:

- uniqueness: Show that $((1), (2)) \Rightarrow \circledast\circledast\circledast\circledast$

  Let X_1, X_2, X_3 be vector fields. By (2)

$$\left.\begin{array}{rl} 0 = (\nabla g)\left(X_i, X_{i+1}, X_{i+2}\right) &= \left(\nabla_{X_{i+2}} g\right)\left(X_i, X_{i+1}\right) \\ &= X_{i+2}\, g(X_i, X_{i+1}) - g\left(\nabla_{X_{i+2}} X_i, X_{i+1}\right) - g\left(\nabla_{X_{i+2}} X_{i+1}, X_i\right)\end{array}\right\} (\#)_i$$

Take $(\#)_{i+1} + (\#)_{i+2} - (\#)_i$:

$$\begin{aligned} 0 &= X_i g(X_{i+1}, X_{i+2}) + X_{i+1} g(X_{i+2}, X_i) - X_{i+2} g(X_i, X_{i+1}) \\ &- g\left(\nabla_{X_i} X_{i+1} + \nabla_{X_{i+1}} X_i, X_{i+2}\right) \\ &- g\left(\nabla_{X_{i+1}} X_{i+2} - \nabla_{X_{i+2}} X_{i+1}, X_i\right) \\ &+ g\left(\nabla_{X_{i+2}} X_i + \nabla_{X_i} X_{i+2}, X_{i+1}\right) \\ &= \text{what we want} \end{aligned}$$

- existence: ⊛ ⊛ ⊛⊛ defines $\nabla_X Y$ (because $(\cdot, \cdot)$ is non-degenerate)

  show

  a) $\nabla_X Y$ is a connection
  b) $T = 0$
  c) $\nabla g = 0$

  b) $2(\nabla_X Y - \nabla_Y X, Z) = 2\left([X, Y], Z\right)$
  $\Rightarrow [X, Y] = \nabla_X Y - \nabla_Y X$

  c) $\nabla g = 0 \Leftrightarrow (\#)_i$
  Definition of $\nabla \Leftrightarrow$ ⊛ ⊛ ⊛⊛ $(\Leftrightarrow (\#)_{i+1} + (\#)_{i+2} - (\#)_i)$
  Take ⊛ ⊛ $⊛⊛_{i+1}$ + ⊛ ⊛ $⊛⊛_{i+2}$ $\Leftrightarrow (\#)_{i+2} + (\#)_i - (\#)_{i+1} + (\#)_i + (\#)_{i+1} - (\#)_{i+2} = 2(\#)_i$

In a chart: Christoffel symbols of the Riemann connection

$$\Gamma^i{}_{lk} = \frac{1}{2} g^{ij} (g_{lj,k} + g_{kj,l} - g_{lk,j})$$

because $\Gamma^i{}_{lk} = \langle e^i, \nabla_{e_l} e_k \rangle = (e^i, \nabla_{e_l} e_k)$
Let $X = e_l = \frac{\partial}{\partial x^l}$, $Y = e^k$, $Z = e_j = g_{ij} e^i$
$$2\left(\nabla_{e_l} e_k, g_{ij} e^i\right) = g_{kj,l} + g_{jl,k} - g_{lk,j}$$

3.1 Geodesic

Definition: a parametrized curve $x(\lambda)$ on M is a geodesic if it solves the variational principle

$$\delta \int_{(1)}^{(2)} d\lambda (\dot{x}, \dot{x}) = 0, \qquad \dot{x} = \frac{dx}{d\lambda}$$

with fixed endpoints (both in $M \ni x, \mathbb{R} \ni \lambda$)

In a chart these are the Euler-Lagrange-Equations for the Lagrangian

$$L(x, \dot{x}) = \frac{1}{2} (\dot{x}, \dot{x}) = \frac{1}{2} g_{ik}(x) \dot{x}^i \dot{x}^k$$

They are

$$
\begin{aligned}
0 = \frac{d}{d\lambda}\frac{\partial \mathcal{L}}{\partial \dot{x}^j} - \frac{\partial \mathcal{L}}{\partial x^j} &= \frac{d}{d\lambda}\left(g_{lj}(x)\dot{x}^l\right) - \frac{1}{2} g_{lk,j}\dot{x}^l\dot{x}^k \\
&= g_{lj,k}\dot{x}^k\dot{x}^l + g_{lj}\ddot{x}^l - \frac{1}{2} g_{lk,j}\dot{x}^l\dot{x}^k \\
&= \frac{1}{2}\left(g_{lj,k} + g_{kj,l}\right)\dot{x}^k\dot{x}^l + g_{lj}\ddot{x}^l - \frac{1}{2} g_{lk,j}\dot{x}^l\dot{x}^k \\
&= g_{lj}\ddot{x}^l + \frac{1}{2}\left(g_{lj,k} + g_{kj,l} + g_{glk,j}\right)\dot{x}^l\dot{x}^k
\end{aligned}
$$

$$
\Longleftrightarrow \quad \ddot{x}^i + \Gamma^i{}_{lk}\dot{x}^l\dot{x}^k = 0 \quad \underline{\text{geodesic equation}} \quad \Leftrightarrow \quad \nabla_{\dot{X}}\dot{X} = 0 \boxtimes
$$

Recall: $X(t) \in T_{\gamma(t)}$ is parallel transported along $\gamma(s) = x(s)$ if

$$
\dot{X}^i + \Gamma^i{}_{lk}\dot{\gamma}^l X^k = 0
$$

$\boxtimes$ says that $\dot{x} = X$ is transported in its own direction.

Remarks:

1. $L = \frac{1}{2}(\dot{x},\dot{x}) = L(x,\dot{x})$ does not depend on λ. Hence

$$
\underbrace{p_i\dot{X}^i}_{\dot{x}^i\frac{\partial L}{\partial \dot{x}^i}} - L = 2L - L = L
$$

is conserved along a geodesic

2. Reparametrization: $\lambda \mapsto \lambda'$
$\dot{x} = x'\frac{d\lambda'}{d\lambda}$, $\ddot{x} = x''\left(\frac{d\lambda'}{d\lambda}\right)^2 + x'\frac{d^2\lambda'}{d\lambda^2}$
Equation $\boxtimes$ is invariant under reparametrization if

$$
\frac{d^2\lambda'}{d\lambda^2} = 0 \;\Leftrightarrow\; \lambda' = a\lambda + b \qquad (a, b \in \mathbb{R}\text{ constants})
$$

$\Rightarrow$ Only linear inhomogenous reparametrizations are admissible.

Such a reparametrization is called affine parameter

3. Assume g is positive definite.
Length of curve

$$
\int_{(1)}^{(2)} d\lambda\,(\dot{x},\dot{x})^{1/2} = \int_{(1)}^{(2)} d\lambda f(L) \quad f = \sqrt{}
$$

is invariant under arbitrary reparametrizations.

Along a geodesic:

$$
\begin{aligned}
\delta\int_{(1)}^{(2)} d\lambda f(L) &= \int_{(1)}^{(2)} d\lambda f'(L)\delta L \\
&= f'(const)\,\delta\int_{(1)}^{(2)} d\lambda L
\end{aligned}
$$

$\to$ geodesic makes length stationary.

$\int_{(1)}^{(2)} d\lambda(\dot{x}, \dot{x})$ is applicable even if g is not positive definite.

Properties of the Riemann connection:

a) inner product of vectors is conserved under parallel transport:

$$(X(t), Y(t))_{\gamma(t)} = (X, Y)_{\gamma(0)}$$

where $\left.\begin{array}{l} X(t) = \tau(t,0)X \\ Y(t) = \tau(t,0)Y \end{array}\right\} \in T_{\gamma(t)} \qquad X, Y \in T_{\gamma(0)}$

because of $\nabla g = 0$, $\quad g_{\gamma(t)} = \tau(t,0) g_{\gamma(0)}$

$$\begin{aligned} (X(t), Y(t))_{\gamma(t)} &= \left(\tau(t,0) g_{\gamma(0)}\right) \left(\tau(t,0)X, \tau(t,0)Y\right) \\ &= g_{\gamma(0)}(X, Y) = (X, Y)_{\gamma(0)} \end{aligned} \qquad \text{cf. Remark (1)}$$

b) covariant differentiation commutes with raising & lowering indices

$$T^i{}_{k;l} = \left(g_{km} T^{im}\right)_{;l} = g_{km} T^{im}{}_{;l}$$

because $g_{km;l} = 0$ For short:

$$\nabla \circ g = g \circ \nabla$$

where g is the "lowering the index"

c) Curvature tensor: symmetries:

i) $(W, R(X,Y)Z) = -\,(Z, R(X,Y)W)$

ii) $(W, R(X,Y)Z) = (X, R(W,Z)Y)$

Proof: $\langle W, R(X,Y)Z\rangle = -\langle R(X,Y)W, Z\rangle$

Set $\omega = gW$

$(\omega, R(X,Y)Z) = -\langle R(X,Y)gW, Z\rangle = -\,(R(X,Y)W, Z)$

ii) says $(W, R(X,Y)Z)$ is symmetric $(X,Y) \leftrightarrow (W,Z)$

$$\begin{aligned} (W, R(X,Y)Z) &= -\,(W, R(Y,Z)X) - (W, R(Z,X)Y) \\ &= (Z, R(Y,W)X) + (Z, R(W,X)Y) \\ 2\,(W, R(X,Y)Z) &= (W, R(Z,Y)X) + (W, R(X,Z)Y) + (Z, R(Y,W)X) \\ &\quad + (Z, R(W,X)Y) \end{aligned}$$

using (i)

Summary:

$$R^i{}_{jkl} = -R^i{}_{jlk} \qquad \text{always}$$

$$\left.\begin{aligned} \sum_{\substack{\text{cyclic permut.}\\ \text{of } (jkl)}} R^i{}_{jkl} = 0 \\ \sum_{(klm)} R^i{}_{jkl;m} = 0 \end{aligned}\right\} \underline{\text{torsion vanishes}}$$

$$\left.\begin{aligned} R_{ijkl} &= -R_{jikl} \\ R_{ijkl} &= R_{klij} \end{aligned}\right\} \text{Riemann connections}$$

note:

$$\begin{aligned} R^i{}_{jkl} &= \langle e^i, R(e_k, e_l)e_j\rangle \\ &= (\tilde{e}^i, \dots) \\ R_{ijkl} &= (e_i, R(e_k, e_l)e_j) \end{aligned}$$

d) Ricci and Einstein tensors

Definition:

$R_{ik} = R^j{}_{ijk}$ (Ricci tensor)

$R = R^i{}_i$ (scalar curvature)

$G_{ik} = R_{ik} - \frac{1}{2}Rg_{ik}$ (Einstein tensor)

We have (∇: Riemann connection):

- $R_{ik} = R_{ki}$
- $R_i{}^k{}_{;k} = \frac{1}{2}R_{;i}$ contracted 2nd Bianchi
- $G_i{}^k{}_{;k} = 0$

Remark: other contractions do not produce new stuff

$$R^i{}_{jki} = -R^i{}_{jik} = -R_{jk}$$

$$R^i{}_{ikl} = -R^i{}_{kli} - R^i{}_{lik} = R_{kl} - R_{lk} = 0$$

Proof: $R_{ik} = g^{jl}R_{lijk} = \underset{\underset{g^{lj}}{\shortparallel}}{g^{jl}} R_{jkli} = R^l{}_{kli} = R_{ki}$

-2nd Bianchi : $R^i{}_{jkl;m} + R^i{}_{jlm;k} + R^i{}_{jmk;l} = 0$

trace(ik): $R^j{}_{jl;m} + \underbrace{R^i{}_{jlm;i}}_{g^{ik}R_{kjlm;i}} - R^j{}_{jm;l} = 0$

Normal coordinates

Recall from linear algebra:
A symmetric non-degenerate bilinear form g_{ik} ($\to$ metric at one point $p \in M$) can be put in normal form:

$$\bar{g}_{lm} = \phi_l{}^i\phi_m{}^k g_{ik} = \eta_{lm} = \underbrace{\text{diag}(\pm 1, \dots, \pm 1)}_{\text{signarure of } g_{ik}} \qquad (\bar{e}_l = \phi_l{}^m e_m)$$

Pseudo-Riemann manifold, connected

g_p: metric at $p \in M$ $\rightarrow$ signature of g_p is constant

Change of coordinates: ${\phi_l}^i = \frac{\partial x^i}{\partial \bar{x}^l}$: $\quad \bar{g}_{lm} = \frac{\partial x^i}{\partial \bar{x}^l}\frac{\partial x^k}{\partial \bar{x}^m} g_{ik}$

However, as a rule: in <u>no</u> chart:

$$\bar{g}_{lm}(\bar{x}) \equiv \eta_{lm}$$

<u>Theorem:</u> In some neighbourhood of any point $p \in M$ there is a chart such that

i) $x^i = 0$ at p

ii) $g_{ij}(0) = \eta_{ij}$

iii) $g_{ij,l}(0) = 0 \quad (\Leftrightarrow {\Gamma^i}_{jl}(0) = 0)$

<u>Remarks:</u>

1) "$\Rightarrow$": $\surd$

"$\Leftarrow$": $0 \underset{\nabla g = 0}{\overset{\uparrow}{=}} g_{ik;l} = g_{ik,l} - {\Gamma^r}_{li} g_{rk} - {\Gamma^r}_{lk} g_{ir}$

2) $g_{ij,lm}(0) = 0$ impossible (as a rule) because ${R^i}_{jkl} \neq 0$

3) ${T^i}_{kl} = {\Gamma^i}_{kl} - {\Gamma^i}_{lk}$ but $T = 0$

4 Time, space and relativity

1. The classical relativity principle

   - clocks, rigid rods } determine the frames of reference in classical physics
   - simultaneity is absolute, geometry is Euclidean } prior geometry
   - free particle (i.e. far away from anything else) are at basics of dynamics
     - (1st law) <u>Inertial frames:</u> trajectory $x(t)$ of a free particle obeys $\ddot{\vec{x}} = 0$
     - (2nd law) <u>Forces:</u> deviation from free motion

   $$m_i \ddot{\vec{x}}_i = \vec{F}_i(\vec{x}_1, \ldots, \vec{x}_N) \qquad m_i: \text{ inertial masses } i = 1, \ldots, N$$

   <u>Examples:</u>

   (a) Particle in electromagnetic field $\vec{E}$:

   $$\vec{F} = e\vec{E}$$

(b) Particle in gravitational field $\vec{g}$:

$$\vec{F} = \tilde{m}\vec{g} \qquad (\tilde{m}: \text{ gravitational mass})$$

Fact:

$$m = \tilde{m}, \qquad \text{hence}$$
$$\not{m}\ddot{\vec{x}} = \not{\tilde{m}}\vec{g}$$

$\Rightarrow$ All free falling particles fall with $\ddot{\vec{x}} = \vec{g}$

Remark: Inertial forces proportional to inertial mass (Scheinkräfte) (dissapear when moving to inertial frame)

2. The Einstein equivalence principle (EP)

   Put free falling particles at the basis of dynamics

   $\Rightarrow$ Gravitational force is an inertial force

   EP (1911) "All freely falling, non-rotating local reference frames are equivalent w.r.t.
   local inertial frame (LIF)
   all local experiments"

   Remarks: 1) non-rotating means no Coreolis-Force
   2) EP is heuristic, to be made precise
   3) valid for all of physics, not just mechanics

   Application: gravitational red-shift

3. The postulates of GR (1915) (extended and clarify EP)

   (a) Time and Space are a 4-dimensional pseudo-Riemannian Manifold M with metric g of signature $(+1, -1, -1, -1)$
   ($p \in M \Leftrightarrow$ events)
   (chart $\Leftrightarrow$ reference frame)
   g expresses measurements done by means of ideal clocks & rods

   (b) Physical quantities/laws are tensors/equations among tensors

   (c) With the exception of g, physical laws only contain quantities already present in SR.

   (d) In a LIF around $p \in M$ physical laws are the same as in SR.
   ↓ normal coordinates

   Remarks: about (a): ideal clock: world line $x(\lambda)$ measures $\Delta\tau$

   $$c^2(\Delta\tau)^2 = g(\dot{x}, \dot{x})(\Delta\lambda)^2$$

   ideal rod: world line of endpoints of rod

   $$g(\dot{x}, \Delta x) = 0$$

measures length Δl: $(\Delta l)^2 = -g(\Delta x, \Delta x)$

in particular coordinates:

$x = (x^0, \dots, x^3)$ such that

world line of clock $x = (ct, 0, 0, 0)$

$\dot{x} = (c, 0, 0, 0)$

about (b): physical quantities in a reference frame are given as components of tensors

- different in each frame
- laws including them are the same in all frames (general form covariance)

about (d): $\Leftrightarrow$ EP

4. Transition SR $\longrightarrow$ GR

(a) laws of inertia

SR	GR
$\ddot{x}^\mu = 0$	$\ddot{x}^\mu + \Gamma^\mu{}_{\nu\sigma}\dot{x}^\nu \dot{x}^\sigma = 0$ ⊠⊠ $\Leftrightarrow \nabla_{\dot{X}} \dot{X} = 0$
$(\dot{x}, \dot{x}) = c^2$	$g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu = (\dot{x}, \dot{x}) = c^2$

free particle

$x^\mu(\tau)$ world line

$\dot{} = \frac{d}{d\tau}$ τ: proper time

⊠⊠ describes gravitational force as

$$\ddot{x}^\mu = -\Gamma^\mu{}_{\nu\sigma}\dot{x}^\nu \dot{x}^\sigma \longrightarrow \Gamma^\mu{}_{\nu\sigma} (\text{not } g_{\mu\nu} \text{ describes gravitational field})$$

$\Gamma^\mu{}_{\nu\sigma}$ ↓ can be transformed away; $g_{\mu\nu}$ ↓ cannot

(b) light rays

SR	GR
$\ddot{x}^\mu = 0$	$\ddot{x}^\mu + \Gamma^\mu{}_{\nu\sigma}\dot{x}^\nu \dot{x}^\sigma = 0$
$(\dot{x}, \dot{x}) = 0$	$(\dot{x}, \dot{x}) = 0$ (null geodesic!)
	includes deflection of light

More generally: Covariant formulation of Maxwell's equations (ME)

e.m. field tensor $F_{\mu\nu} = -F_{\nu\mu}$

In an inertial frame in the sense of SR

$$F_{\mu\nu} = \begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ & 0 & -B_3 & B_2 \\ & & 0 & -B_1 \\ & & & 0 \end{pmatrix}$$

- homogeneous ME

SR	GR
$F_{\mu\nu,\sigma} + \text{cycl.} = 0$	$\underset{\text{covariant derivative}}{F_{\mu\nu;\sigma}} + \text{cycl.} = 0$

- inhom. ME

SR	GR
$F^{\mu\nu}{}_{,\mu} = \frac{1}{c} j^\nu$	$F^{\mu\nu}{}_{;\mu} = \frac{1}{c} j^\nu$

Recipe: replace 1[st] order partial derivatives by covariant derivatives
Consequence of ME: charge conservation, i.e. continuity equation

SR	GR
$j^\nu{}_{,\nu} = 0$	$j^\nu{}_{;\nu} = 0$

Better: rederive from ME in GR

$$\frac{1}{c} j^\nu{}_{;\nu} = F^{\mu\nu}{}_{;\mu\nu} = F^{\mu\nu}{}_{;\nu\mu} + \underbrace{R^\mu{}_{\tau\mu\nu}}_{R_{\tau\nu}} F^{\tau\nu} + \underbrace{R^\nu{}_{\tau\mu\nu}}_{R_{\tau\mu}} F^{\mu\tau}$$

not necessary symmetric in cov. derivatives as in the case of partial derivatives

($\mu\tau \to \tau\nu$, $R_{\tau\mu} \to R_{\nu\tau}$)

$$= -\underbrace{F^{\nu\mu}{}_{;\nu\mu}}_{=-F^{\mu\nu}{}_{\mu\nu}} + \underbrace{(R_{\tau\nu} + R_{\nu\tau})}_{=0} F^{\tau\nu}$$

$$= 0$$

Homogeneous ME in SR/GR:

$$F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu} = A_{\nu;\mu} - A_{\mu;\nu} \qquad A^\mu : \text{4-vector potential}$$

(c) Equations of motion of charged particle in an e.m. field

$$x^\mu(\tau) : \text{trajectory} \qquad \tau : \text{proper time}$$

$$(\nabla_{\dot{x}} \dot{x})^\mu = \ddot{x}^\mu + \Gamma^\mu{}_{\nu\sigma} \dot{x}^\nu \dot{x}^\sigma = \frac{e}{mc} F^{\mu\nu}(x) \dot{x}_\nu$$

come from a variational principle:

$$\delta \int_{(1)}^{(2)} d\tau \left(c^2 + \frac{e}{mc} (\dot{x}, A) \right) = 0 \qquad \text{Fixed endpoints in space time, } \tau \text{ not fixed}$$

5. Geodesic equation $\longrightarrow$ Newtonian free fall

$\underset{\ddot{x}^\mu+\Gamma^\mu{}_{\nu\sigma}\dot{x}^\nu\dot{x}^\sigma}{\uparrow}$ $\qquad$ $\underset{\ddot{\vec{x}}=-\vec{\nabla}\varphi}{\uparrow}$

Newton's equations of motion emerges as an approximamion for:

- slow particles
- in coordinates representing times & lengths in a small neighbourhood of $\vec{x}=0$

$$g_{\mu\nu}(x)=\eta_{\mu\nu} \qquad \text{for } x=(ct,\underbrace{0,0,0}_{\vec{x}})$$

$$\eta_{\mu\nu}=\begin{pmatrix}1&&&\\&-1&&\\&&-1&\\&&&-1\end{pmatrix}$$

Trajection within a region space time:

$$\begin{aligned}g_{\mu\nu}(x)&=\eta_{\mu\nu}+h_{\mu\nu}(x)\\|h_{\mu\nu}|&<<1\end{aligned} \qquad \text{"weak gravitational field"}$$

Hence $h_{\mu\nu,0}=0$ $\qquad$ at $\vec{x}=0$

$$\begin{aligned}c^2=(\dot{x},\dot{x})&=\eta_{\mu\nu}\dot{x}^\mu\dot{x}^\nu+\mathcal{O}(h)\\&=(c^2-\vec{v}^2)\left(\frac{dt}{d\tau}\right)^2+\mathcal{O}(h) \qquad x^\mu(\tau)=(ct,\vec{x})\big|_{t=t(\tau)}\end{aligned}$$

$\Rightarrow \frac{d}{d\tau}=\frac{d}{dt}$ $\qquad$ up to $\mathcal{O}(v^2)+\mathcal{O}(h)$

$\longrightarrow$ $\underline{\dot{x}^\mu=(c,\vec{v})}$

- At first, particle (almost) at rest $\qquad$ $\mathcal{O}(v)=0:\dot{x}^\mu=(c,\vec{0})$

$$\ddot{x}^i=c^2\Gamma^i{}_{00} \qquad i=1,2,3$$

with

$$\begin{aligned}\Gamma^i{}_{00}&=\frac{1}{2}\eta^{ik}(h_{0k,0}+h_{0k,0}-h_{00,k})\\&=-h_{0i,0}+\frac{1}{2}h_{00,i}\\&=\frac{1}{2}h_{00,i}\end{aligned}$$

Hence

$$\ddot{\vec{x}}=-\vec{\nabla}\varphi \qquad \text{with } \varphi=\frac{1}{2}c^2h_{00}$$

$$g_{00}=1+\frac{2\varphi}{c^2} \qquad \varphi:\ \text{Newton's gravitational potential}$$

- Keep terms $\propto \vec{v}$. Then

$$\ddot{x}^i = -c^2 {\Gamma^i}_{00} - 2c{\Gamma^i}_{0j}\dot{x}^j \qquad ⊠⊠⊠$$

Because $+{\Gamma^i}_{j0}\dot{x}^j$: Γ is symmetric in lower indices
with

$$\begin{aligned} {\Gamma^i}_{0j} &= \frac{1}{2}\eta^{ik}(h_{0k,j} + h_{jk,0} - h_{0j,k}) \\ &= \frac{1}{2}((h_{0j,i} - h_{0i,j}) \qquad \text{at } \vec{x} = 0 \end{aligned}$$

Since $\vec{x} \cong \vec{v}t$ keep terms $\propto \vec{x}$ in ${\Gamma^i}_{00}$
For comparison: Newtonian description

$$\ddot{\vec{x}} = \vec{g} - \underbrace{\vec{a}}_{\text{Führungsbeschl.}} - \underbrace{2\vec{\omega} \wedge \dot{\vec{x}}}_{\text{coriolis force}} - \underbrace{\vec{\omega} \wedge (\vec{\omega} \wedge \vec{x})}_{\text{centrifugal force}} - \dot{\vec{\omega}} \wedge \vec{x} \qquad ⊠⊠⊠⊠$$

(⊠⊠⊠), (⊠⊠⊠⊠) agree for

$$\begin{aligned} g_{00} &= 1 + \frac{2}{c^2}\left(\varphi - (\vec{\omega} \wedge \vec{x})^2\right) = 1 + h_{00} \\ g_{0i} &= h_{0i} = -\frac{1}{c}\left(\vec{\omega} \wedge \vec{x}\right)_i = -\frac{1}{c}\epsilon_{ijk}\omega_j x_k \end{aligned}$$

Indeed

$$\begin{aligned} 2c{\Gamma^i}_{0j} &= c(h_{0j,i} - h_{0i,j}) = 2\epsilon_{jik}\omega_k \\ 2c{\Gamma^i}_{0j}\dot{x}^j &= 2(\vec{\omega} \wedge \dot{\vec{x}})_i \\ \vec{\omega} \wedge (\vec{\omega} \wedge \vec{x}) &= -\frac{1}{2}\vec{\nabla}\left(\vec{\omega} \wedge \vec{x}\right)^2 \\ c^2 h_{0i,0} &= -(\dot{\vec{\omega}} \wedge \vec{x})_i \end{aligned}$$

6. Physical meaning of curvature

We feel gravitational force from earth, but not from sun of moon, since we're in free fall w.r.t. them. But the EP is not true when "going global", for example we have tides, the moon doesn't act the same an the two sides of the earth.

The physical meaning of curvature is that of a relative acceleration of nearby freely falling particles.

First consider Newtionan mechanics/SR: free particles in an inertial frame

$$\frac{d^2\vec{u}}{dt^2} = 0$$

Now consider GR:

family of geodesics

$$\left.\begin{aligned} x(\tau)\text{with 4-velocity } u(x) : \tfrac{dx}{d\tau} &= u\left(x(\tau)\right) \\ (u, u) &= c^2 \end{aligned}\right\} \nabla_u u = 0$$

x are orbits of vector field u parametrized by proper time τ

φ_τ flow generated by u

We want to understand the relative displacement, starting form nearby particles at points p, q in $\{\tau = 0\}$

$\{\tau = 0\} \ni p, q \mapsto \varphi_\tau(p), \varphi_\tau(q)$

$\{\tau = 0\} \supset \gamma \mapsto \varphi_{\tau \circ \gamma}$

vector field n in $\{\tau = 0\}$, $n = \frac{d\gamma}{ds}$

$$n_p \longmapsto \varphi_{\tau *} n_p =: n_{\varphi_\tau(p)} \qquad \text{(Lie transport)}$$

Now from the definition of the Lie derivative:

$$0 = L_u n := [u, n] \quad \nwarrow (L_u p)_p = \frac{d}{d\tau}\left(\varphi_\tau{}^* n_{\varphi_\tau(p)}\right)|_{\tau=0}$$

i.e. u and n commute.

Hence, by torsion = 0:

$$\underbrace{\nabla_u n}_{\substack{\text{relative}\\ \text{velocity}}} = \nabla_n u$$

but we're interested in the relative acceleration, therefore we compute

$$\nabla_u^2 n = \nabla_u \nabla_n u = [R(u, u) + \nabla_n \underbrace{\nabla_u}_{\nabla_u u = 0}]u$$

$$= R(u, u)u$$

Thus the relative acceleration of nearby particles is given by

$$\nabla_u^2 n = R(u, u)u \qquad \text{equation of geodesic deviation}$$

Curvature manifests itself through relative acceleration of nearby freely falling particles (tidal forces)

<u>Remarks:</u>

(a) Suppose u is perpendicular to $\{\tau = 0\}$, i.e.

$$g(u, n) = 0$$

Then this holds everywhere, because

$$u\,[g(u, n)] = \nabla_u\,[g(u, n)] = (\underbrace{\nabla_u g}_{\substack{=0,\\ \nabla g = 0}})(u, n) + g(\underbrace{\nabla_u u}_{\substack{=0,\\ \text{geodesic}}}, n) + g(u, \underbrace{\nabla_u n}_{= \nabla_n u})$$

$$= \tfrac{1}{2} n [\underbrace{g(u, u)}_{= c^2}] = 0$$

(b) Let e_μ be a basis of vector fields with $e_0 = u$
Then

$\langle e^i, \nabla_n^2 e_i \rangle$: relative acceleration in direction $i = (1,2,3)$ of particles in direction i

From the equation of geod. dev.

$$\sum_{i=1}^{3} \langle e^i, \nabla_u^2 e_i \rangle = \sum_{i=1}^{3} \langle e^i, R(u,e_i)u \rangle \overset{\substack{R(u,n)\\=0\\ \downarrow}}{=} \langle e^\mu, R(u,e_\mu)u \rangle = -\operatorname{Ric}(u,u)$$

This is the end of a chapter on gravitation in an external gravitational field.
The next chapter is even more important, namely deals with gravitational field itself.

5 The Einstein field equations

5.1 The energy-momentum tensor

SR: Energy-momentum vector p^μ of a particle:

$$p^\mu = \left(\frac{E}{c}, \vec{p}\right) = \frac{m}{\sqrt{1-\frac{v^2}{c^2}}}(c, \vec{v})$$

m: (rest) mass
E: energy
$\vec{p}$: momentum

We want to generalize this to a field. The basic thing is that a particle is somewhere, but a field is everywhere.

Field: Energy-momentum tensor $T^{\mu\nu}$

T^{00}: energy density
T^{0k}: $\frac{1}{c}\cdot$ energy current density (in direction k)
T^{i0}: $c\cdot$momentum density (i[th] component)
T^{ik}: momentum current density
(momentum ↑ i[th] component; current ↑ k[th] direction)

That is:

$T_{00} d^3x$: energy in d^3x
$T_{i0} d^3x$: $c\cdot$momentum in d^3x

$$\left.\begin{array}{ll} \sum_{k=1}^{3} T^{0k} d\sigma_k : & \frac{1}{c}\text{energy flow} \\ \sum_{k=1}^{3} T^{ik} d\sigma_k : & \text{momentum flow} \end{array}\right\} \text{from side 1 to side 2}$$

- Symmetry: $T^{\mu\nu} = T^{\nu\mu}$

- Isotropy: If, in some frame, $T^{\mu\nu}$ is

$$\text{invariant under rotations } \Lambda = \left(\begin{array}{c|ccc} 1 & 0 & 0 & 0 \\ \hline 0 & & & \\ 0 & & R & \\ 0 & & & \end{array}\right), \ R \in SO(3), \text{ then}$$

$$T^{\mu\nu} = \left(\begin{array}{c|ccc} \rho c^2 & 0 & 0 & 0 \\ \hline 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{array}\right), \ \rho c^2\text{: energy density, } p\text{: pressure}$$

- Energy-momentum conservation (for a free field)

$$T^{\mu\nu}{}_{,\nu} = 0$$

$$\rightarrow \tfrac{d}{dt} \underbrace{\int_{x^0=t} d^3x T^{\mu 0}}_{\text{total 4-momentum}} = 0$$

(analog in ED: $j^{\mu}_{,\mu} = 0 \rightarrow \frac{d}{dt}\int_{x^0=t} j^0 d^3x = 0$)

<u>In GR:</u> freely falling field

$$T^{\mu\nu}_{;\nu} = 0$$

Models:

1. <u>e.m. field</u> $\quad T^{\mu\nu} = F^{\mu}{}_{\sigma}T^{\sigma\nu} - \frac{1}{4}\left(F_{\rho\sigma}F^{\sigma\rho}\right)g^{\mu\nu}$

   trace: $\quad T^{\mu}{}_{\mu} = 0 \qquad g^{\mu}{}_{\mu} = \delta^{\mu}{}_{\mu} = 4$

$$T^{\mu\nu} = \left(\begin{array}{c|c} \frac{1}{2}(\vec{E}^2 + \vec{B}^2) & \vec{E} \wedge \vec{B} \\ \hline \vec{E} \wedge \vec{B} & \frac{1}{2}(\vec{E}^2 + \vec{B}^2)\delta_{ik} - E_iE_k - B_iB_k \end{array}\right)$$

   If on average $T^{\mu\nu}$ is isotropic

$$T^{\mu}{}_{\mu} = \underset{\substack{\uparrow \\ \text{because of lowering the indices}}}{\rho c^2 - 3p} = 0 \qquad \Longrightarrow \qquad \underset{\text{equation of state}}{p = \frac{1}{3}\rho c^2}$$

   Further models of matter (particles, in a continuum description):

2. <u>Dust</u> ("cold dark matter"): Swarm of particles with a <u>common</u> local velocity

   $\rho(x)$: mass density in a local rest frame (LIF)
   (a scalar, $\bar{\rho}(\bar{x}) = \rho(x)$, by definition)
   $u^{\mu}(x)$: 4-velocity

In a local rest frame

$$T^{\mu\nu} = \left(\begin{array}{c|c} \rho c^2 & 0 \\ \hline 0 & 0 \end{array}\right) \qquad u^\mu = \begin{pmatrix} c \\ 0 \end{pmatrix} *$$

In general

$$T^{\mu\nu} = \rho u^\mu u^\nu$$

because:

- both sides are tensors
- in rest frame: agrees with *

Similary: current density $j^\mu = \rho u^\mu$

<u>equations of motion:</u>

- particle conservation:
  SR: ${j^\mu}_{,\mu} = 0$
  GR: ${j^\mu}_{;\mu} = (\rho u^\mu)_{;\mu} = 0$ continuity equation
- free fall

$$\nabla_u u = 0 \qquad \begin{pmatrix} \dot{x}(\tau) = u(x(\tau)) \\ \nabla_{\dot{x}}\dot{x} = 0 = \nabla_u u \end{pmatrix}$$

This implies

$${T^{\mu\nu}}_{;\nu} = 0$$

Indeed:

$${T^{\mu\nu}}_{;\nu} = u^\mu \underbrace{(\rho u^\nu)_{;\nu}}_{\substack{=0,\\ \text{continuity}\\ \text{relation}}} + \rho \underbrace{u^\nu {u^\mu}_{;\nu}}_{\substack{(\nabla_u u)^\mu = 0\\ \text{free fall}}}$$

$$\implies T^{\mu\nu} \text{ is divergence free}$$

Conversely: $T^{\mu\nu}$ and $u^\mu u_\mu = c^2$ implies equations of motion since

$$0 = u_\mu {T^{\mu\nu}}_{;\nu} = \underbrace{u_\mu u^\mu}_{=c^2}(\rho u^\nu)_{;\nu} + \rho u^\nu \underbrace{u_\mu {u^\mu}_{;\nu}}_{\frac{1}{2}(\underbrace{u_\mu u^\mu}_{=c^2})_{;\nu}=0}$$

$$\implies (\rho u^\nu)_{;\nu} = 0 \qquad \Rightarrow \nabla_u u = 0$$

3. Ideal fluid: Swarm of particles with mean local velocity; velocity distribution is isotropic in rest frame of distribution (where mean vel.= 0)

$$\begin{aligned} \rho(x)c^2 &: \text{energy density} \\ p(x) &: \text{pressure} \\ u^\mu &: \text{mean local 4-vector} \end{aligned}$$

in a local rest frame.

Classical (Newtonian) equations of motion:

$$\left.\begin{array}{l} \frac{\partial\rho}{\partial t} + \div(\rho\vec{v}) = 0 \\ \rho\left(\left(\frac{\partial\vec{v}}{\partial t}\right) + (\vec{v}\cdot\vec{\nabla})\vec{v}\right) = -\vec{\nabla}p \end{array}\right\} \text{Euler's equations}$$

In a local rest frame:

$$T^{\mu\nu} = \left(\begin{array}{c|ccc} \rho c^2 & & 0 & \\ \hline & p & 0 & 0 \\ 0 & 0 & p & 0 \\ & 0 & 0 & p \end{array}\right) \quad **$$

In general:

$$T^{\mu\nu} = \left(\rho + \frac{p}{c^2}\right)u^\mu u^\nu - pg^{\mu\nu}$$

- tensors
- reduces to ** in a local rest frame

Equations of motion: Starting point ${T^{\mu\nu}}_{;\nu} = 0$

$$\begin{aligned} {T^{\mu\nu}}_{;\nu} &= u^\mu\left(\left(\rho + \frac{p}{c^2}\right)u^\nu\right)_{;\nu} + \left(\left(\rho + \frac{p}{c^2}\right)u^\nu\right){u^\mu}_{;\nu} - p_{;\nu}g^{\mu\nu} \\ &\stackrel{!}{=} 0 \end{aligned}$$

Hence:

$$u_\mu {T^{\mu\nu}}_{;\nu} = \cancel{c^2}\left(\left(\rho + \frac{p}{c^2}\right)u^\nu\right)_{;\nu} - \frac{p_{;\nu}}{c^2}u^\nu = 0$$

$$\left(g_{\sigma\mu} - \frac{u_\sigma u_\mu}{c^2}\right){T^{\mu\nu}}_{;\nu} = \left(\rho + \frac{p}{c^2}\right)(\nabla_u u)_\sigma - p_{;\sigma} + \frac{u_\sigma u^\nu}{c^2}p_{;\nu} = 0$$

Classical limit ($|\vec{v}| \ll c$, i.e. $u^\mu = (c, \vec{v})$)

in free fall (${\Gamma^\alpha}_{\beta\gamma} = 0$)

$$\frac{\partial\rho}{\partial t} + \frac{\partial}{\partial t}\left(\frac{p}{c^2}\right) + \div(p\vec{v}) + \frac{p}{c^2}\div\vec{v} = 0$$

$$\left(\rho + \frac{p}{c^2}\right)\left(\frac{\partial\vec{v}}{\partial t} + \left(\vec{v}\cdot\vec{\nabla}\right)\cdot\vec{v}\right) + \vec{\nabla}p + \left(\frac{\partial p}{\partial t} + \left(\vec{v}\cdot\vec{\nabla}\right)p\right)\frac{\vec{v}}{c^2} = 0$$

differs from the Euler equation, because: fluid may have particles with relativistic velocity despite $|\vec{v}| \ll c$

reduces to Euler for small particles: $p \ll \rho c^2$

For several fields: total energy-momentum tensor $T^{\mu\nu}$,

$$T^{\mu\nu}{}_{;\nu} = 0$$

5.2 The field equations of Gravitation **FE; Einstein 1915**

$$\underset{\substack{\uparrow \\ \text{Einstein-Tensor}}}{G^{\mu\nu}} = \underset{\text{gravitational constant}}{\kappa} \; \underset{\substack{\nwarrow \\ \text{total en.mom. tensor}}}{T^{\mu\nu}}$$

$$G_{\mu\nu} = \underset{\substack{\uparrow \\ \text{Ricci-Tensor}}}{R_{\mu\nu}} - \frac{1}{2} \underset{\substack{\parallel \\ R^{\mu}{}_{\mu}}}{R} \, g_{\mu\nu}$$

1. "Matter tells how space-time curves": partial differential equations for $g_{\mu\nu}$
2. 2nd Bianchi identity $G^{\mu\nu}{}_{;\nu} = 0$

   implies

   $T^{\mu\nu}{}_{;\nu} = 0$: This is a necessary condition for the field equation to have solutions

   (integrability condition)

   (c.f. $F^{\mu\nu}{}_{;\mu} = \frac{j^\nu}{c} \longrightarrow \underset{\text{needed to solve field eqs. in ED}}{j^\nu{}_{;\nu} = 0}$)

3. In the case of dust: FE$\longrightarrow T^{\mu\nu}{}_{;\nu} = 0 \longrightarrow \nabla_u u = 0$

   "Geometry tells matter how to fall"

4. equivalent writing of FE: trace of FE

   $R - 2R = \kappa T$ $\qquad R = R^{\mu}{}_{\mu}$

   i.e. $R = -\kappa T$ $\qquad T = T^{\mu}{}_{\mu}$

   Hence

   $$R^{\mu\nu} = \kappa \left(T^{\mu\nu} - \frac{1}{2} T g^{\mu\nu} \right)$$

   In particular: if $T = 0$ (e.g. e.m. field)

   $R^{\mu\nu} = \kappa T^{\mu\nu}$

   in vacuum: $R^{\mu\nu} = 0$

5. Mean geodesic deviation relative to geodesic (4-vel. u^μ)

$$\begin{aligned}
-Ric(u,u) &= -R_{\mu\nu}u^\mu u^\nu \\
&= -\kappa(T_{\mu\nu}u^\mu u^\nu - \frac{1}{2}Tc^2) \\
&= -\kappa(\rho c^4 - \frac{1}{2}(\rho c^2 - 3p)c^2) \\
&= -\frac{\kappa c^2}{2}(\rho c^2 - 3p)
\end{aligned}$$

gravity is attractive if $\rho c^2 + 3p > 0$

The Newtonian Limit:

$$\vec{F}_{12} = -G_0 m_1 m_2 \frac{\vec{r}}{r^3}$$

pass to continous mass distribution ρ ($m_1 \rightsquigarrow \rho(\vec{y})d^3y$, $m_2 = m$ at $\vec{x} = 0$)

$$\vec{F} = -m\vec{\nabla}\varphi \qquad \text{with } \varphi(x) = -G\int d^3y \frac{\rho(\vec{y})}{|\vec{x}-\vec{y}|}$$

$$\Delta\frac{1}{|\vec{x}|} = -4\pi\delta(\vec{x})$$

Hence:

$$\Delta\varphi = 4\pi G_0\rho \qquad \text{Poisson equation}$$

Assume

$$\begin{aligned}
g_{\mu\nu} &= \eta_{\mu\nu} + h_{\mu\nu} \qquad |h_{\mu\nu}| \ll 1 \\
h_{\mu\nu,0} &= 0 \qquad (h_{\mu\nu}(t, \vec{x} = 0) = 0) \\
\to \Gamma^i{}_{00} &= \frac{1}{2}h_{00,i} = \frac{\varphi_{,i}}{c^2} \qquad \text{i.e. } h_{00} = \frac{2\varphi}{c^2} \\
R^i{}_{0k0} &= \Gamma^i{}_{00,k} - \underbrace{\Gamma^i{}_{k0,0}}_{=0} + \mathcal{O}(h^2) \\
&= \frac{1}{c^2}\varphi_{,ik} \\
R_{00} &= \frac{1}{c^2}\sum_{i=1}^{3}\varphi_{,ii} = \frac{\Delta\varphi}{c^2}
\end{aligned}$$

Assume further: velocity of matter $|\vec{v}| \ll c$. Then $|T^{ij}| \ll T^{00}$

(e.g. dust:

$$u^i = \gamma v^i \qquad \gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$T^{ij} = \rho u^i u^j = \rho\gamma^2 v^i v^j, \qquad T^{00} = \rho\gamma^2 c^2$$

Then $T \equiv T^\alpha{}_\alpha \cong T^{00} = \rho c^2 \qquad \gamma \approx 1,$ da $|\vec{v}| \ll c$

00-Component of FE:

$$\frac{1}{c^2}\Delta\varphi = \kappa\rho c^2\underbrace{(1 - \frac{1}{2})}_{\frac{1}{2}} \qquad \text{i.e. } \Delta\varphi = \frac{\kappa c^4}{2}\rho$$

$\Longrightarrow$ identify constants:

$$\underbrace{\kappa_0 = \frac{8\pi G_0}{c^4}}_{\substack{\text{agrees with Newtonian Gravity} \\ \text{in case the field is weak } (h_{\mu\nu} \cong 0) \\ \text{\& matter is slow } |\vec{v}| \ll c}}$$

The cosmological Term: **(Einstein 1917)**

$$G^{\mu\nu} - \Lambda g^{\mu\nu} = \kappa_0 T^{\mu\nu}$$

Λ: cosmological constant, relevant today

- consistent with $T^{\mu\nu}{}_{,\nu} = 0$ because $g^{\mu\nu}{}_{,\sigma} = 0$
- LHS of the form $aG^{\mu\nu} + bg^{\mu\nu}$ is the most general expression $D[g]^{\mu\nu}$ which
    - contains derivative of $g_{\mu\nu}$ of order ≤ 2
    - $D[g]^{\mu\nu}{}_{;\nu} = 0$
      (Lovelock's Theorem)
- Rewriting $G^{\mu\nu} = \kappa_0 \left(T^{\mu\nu} + \frac{\Lambda}{\kappa_0} g^{\mu\nu} \right)$

$\rightarrow t^{\mu\nu} = \frac{\Lambda}{\kappa_0} g^{\mu\nu}$ is en.-mom. tensor of vacuum

$$=: \left(\begin{array}{c|ccc} \rho c^2 & 0 & 0 & 0 \\ \hline 0 & p & & \\ 0 & & p & \\ 0 & & & p \end{array} \right) \qquad p = -\rho c^2 = -\frac{\Lambda}{\kappa_0}$$

$\rho c^2 + p^3 = -\frac{2\Lambda}{\kappa_0}$: $\qquad \Lambda > 0$: gravity is repelling

Today: $p = W\rho c^2$ with $W \cong -1$ (dark energy)
Observational data do not prove $W \neq -1$

5.3 The Hilbert action

The FE can be obtained from a form covariant variational principle.
Preliminary: canonical measure associated with $g_{\mu\nu}$: Transition function $x = X(\bar{x})$

$$d^n x = \left| \det \left(\frac{\partial x^i}{\partial \bar{x}^j} \right) \right| d^n \bar{x} \qquad \text{is not invariant}$$

$$\bar{g}_{ij}(\bar{x}) = \frac{\partial x^k}{\partial \bar{x}^i} \frac{\partial x^l}{\partial \bar{x}^j} g_{kl}(x)$$

and

$$g(x) = \det\left(g_{ij}(x)\right)$$

thus

$$\bar{g}(\bar{x}) = \left[\det\left(\frac{\partial x^i}{\partial \bar{x}^j}\right)\right]^2 g(x)$$

But

$$\sqrt{|g|}d^nx = \sqrt{|g|}\left|\det\left(\frac{\partial x^i}{\partial \bar{x}^j}\right)\right| d^n\bar{x} = \sqrt{|\bar{g}|}d^n\bar{x} \qquad \text{is invariant}$$

Action: Let $D \subset M$ (space-time), compact

$$S_D[g] = \int_D R\sqrt{-g}\, d^4x \qquad \text{R: scalar curvature of } (g_{\mu\nu})$$

Property: The Euler-Lagrange equations to S_D are the FE in vacuum:

$$\delta S_D[g] = 0$$

for any variation δg vanishing on ∂D is equivalent to $G_{\mu\nu} = 0$

In fact:

$$\delta S_D[g] = \int_D G_{\mu\nu}\delta g^{\mu\nu}\sqrt{-g}\, d^4x + \underbrace{\int_D W^\alpha{}_{;\alpha}\sqrt{-g}\, d^4x}_{\substack{= \int_{\partial D} W^\alpha\sqrt{-g}\, do_\alpha \\ = 0}}$$

where

$$W^\alpha = g^{\mu\nu}\delta\Gamma^\alpha{}_{\mu\nu} - g^{\alpha\mu}\Gamma^\nu{}_{\nu\mu}$$

Note: $W^\alpha{}_{;\alpha}\sqrt{-g} = (W^\alpha\sqrt{-g})_{,\alpha}$

Proof: $\delta\int_D R\sqrt{-g}\, d^4x = \int_D \delta\left(g^{\mu\nu}R_{\mu\nu}\sqrt{-g}\right) d^4x$

$$= \underbrace{\int_D (\delta R_{\mu\nu})\, g^{\mu\nu}\sqrt{-g}\, d^4x}_{I} + \underbrace{\int_D R_{\mu\nu}\delta\left(g^{\mu\nu}\sqrt{-g}\right) d^4x}_{II}$$

I : $\quad R_{\mu\nu} = \Gamma^\alpha{}_{\mu\nu,\alpha} - \Gamma^\alpha{}_{\mu\nu,\nu} + \Gamma^\rho{}_{\mu\nu}\Gamma^\alpha{}_{\rho\alpha} - \Gamma^\rho{}_{\mu\alpha}\Gamma^\alpha{}_{\rho\nu}$

variation at $p \in M \to x_0$ is normal coordinate

$\delta R_{\mu\nu} = (\delta\Gamma^\alpha{}_{\mu\nu})_{,\alpha} - (\delta\Gamma^\alpha{}_{\mu\alpha})_{,\nu}$

$$= (\delta\Gamma^\alpha{}_{\mu\nu})_{;\alpha} - (\delta\Gamma^\alpha{}_{\mu\alpha})_{;\nu}$$ ←Palatini-Identity holds in any coord. throughout D

But $\delta\Gamma^\alpha{}_{\mu\nu}$ is tensorial

$$g^{\mu\nu}\delta R_{\mu\nu} = (g^{\mu\nu}\delta\Gamma^\alpha{}_{\mu\nu})_{;\alpha} - (g^{\mu\nu}\delta\Gamma^\alpha{}_{\mu\alpha})_{;\nu}$$

(indices relabelled: $\alpha \to \nu$, $\nu \to \alpha$)

$$= W^\alpha{}_{;\alpha}$$

II : linear algebra: a $n \times n$ matrix $A = A(x)$

$\frac{1}{\det A}\frac{d}{d\lambda}\det A = tr\,(A^{-1}\dot{A})$ (follows: $\log\det A = tr\,\log A$)

or else: $\det A = \det(A_1, \dots, A_n)$ A_i : i-th row

$$\Rightarrow \frac{d}{d\lambda}\det A = \sum_{i=1}^{n} \det(A_1, \dots, \dot{A}_i, \dots, A_n)$$

$$= \sum_{\substack{i=1\\j=1}}^{n} \dot{a}_{ij} M_{ij}$$ ← minor (i-th row, j-th column erased)

$$= (A^{-1})_{ji} \det A$$ Cramer's Rule

$$A^{-1}A = 1$$

$$\Rightarrow (A^{-1})^{\cdot} A + A^{-1}\dot{A} = 0$$

$g = \det(g_{ik})$

$\delta g = g g^{\mu\nu}\delta g_{\mu\nu} = -g g_{\alpha\beta}\delta g^{\alpha\beta}$

$(\delta g^{\mu\nu})g_{\nu\sigma} = -g^{\mu\nu}\delta g_{\nu\sigma}$

$\delta\sqrt{-g} = \frac{-1}{2\sqrt{-g}}\delta g = -\frac{1}{2}\sqrt{-g}g_{\alpha\beta}\delta g^{\alpha\beta}$

$$\Rightarrow \delta(g^{\mu\nu})\sqrt{-g} = \sqrt{-g}\delta g^{\mu\nu} - \tfrac{1}{2}g^{\mu\nu}\sqrt{-g}(g_{\alpha\beta}\delta g^{\alpha\beta})$$

$$\Rightarrow II = \sqrt{-g}\left(R_{\mu\nu}\delta g^{\mu\nu} - \tfrac{1}{2}R g_{\alpha\beta}\delta g^{\alpha\beta}\right)$$
$$= \sqrt{-g}\underbrace{(R_{\mu\nu} - \tfrac{1}{2}R g_{\mu\nu})}_{G_{\mu\nu}}\delta g^{\mu\nu}$$

Proof: of $W^\alpha{}_{;\alpha}\sqrt{-g} = (W^\alpha\sqrt{-g})_{,\alpha}$

$$W^\alpha{}_{;\alpha} = W^\alpha{}_{,\alpha} + \Gamma^\alpha{}_{\alpha\mu}W^\mu \qquad \text{with } \Gamma^\alpha{}_{\alpha\mu} = \tfrac{1}{2}g^{\alpha\beta}(g_{\alpha\beta,\mu} + \underbrace{g_{\mu\beta,\alpha} - g_{\alpha\mu,\beta}}_{\substack{\text{antisymmetric in}\\ \text{commuting } \alpha,\beta}})$$

$$(-\sqrt{g})_{,\alpha} = \tfrac{1}{2}\sqrt{-g}g^{\mu\nu}g_{\mu\nu,\alpha} \qquad = \tfrac{1}{2}g^{\alpha\beta}g_{\alpha\beta,\mu}$$

Remarks:

1. $$\delta\int_D \sqrt{-g}\,d^4x = -\tfrac{1}{2}\int_D g_{\alpha\beta}\delta g^{\alpha\beta}\sqrt{-g}\,d^4x$$

   Hence:
   $$\delta\int_D (\tfrac{1}{2}R + \Lambda)\sqrt{-g}\,d^4x = 0 \Longrightarrow G_{\mu\nu} - \Lambda g_{\mu\nu} = 0$$

2. S_D depends on R, and hence on $\partial^2 g$
   Usual actions depend on the fields up to their first derivative.

A variant of Hilbert action of this sort is the Palatini-action

$$S_D[g,\Gamma] = \int_D R\sqrt{-g}\, d^4x$$

where $R = g^{\alpha\beta}R_{\alpha\beta}$ and $R_{\alpha\beta}$ is the Ricci of the symmetric connention Γ independent of g

Then

$$\delta_g S = 0 \quad \Rightarrow G_{\mu\nu} = 0$$
$$\delta_\Gamma S = 0 \quad \Rightarrow \nabla g = 0$$

Include matter: Consider any field $\psi = (\psi_A)$ with action of the form:

$$S_D[\psi, \nabla_g\psi] = \int_D \mathcal{L}(\psi, \nabla_g\psi)\sqrt{-g}\, d^4x$$

where ∇_g is the covariant derivative of $g = (g_{ij})$

$\mathcal{L}$ is invariant under arbitrary diffeomorphisms φ (or change of coordinates)

$\mathcal{L}(\varphi^*\psi, \nabla_{\varphi^* g}\varphi^*\psi) = \varphi^*\mathcal{L}(\psi, \nabla_g\psi)$

$= \mathcal{L}(\quad,\quad) \circ \varphi$

The Euler-Lagrange equations $\delta_\psi S_D = 0$ and

$\frac{\partial\mathcal{L}}{\partial\psi_A} - \nabla_\mu \frac{\partial\mathcal{L}}{\partial(\nabla_\mu\psi_A)} = 0$

A symmetric energy-momentum tensor is defined through

$$\delta_g \int_D \mathcal{L}(\psi, \nabla_g\psi)\sqrt{-1}\, d^4x := -\frac{1}{2}\int_D T^{\mu\nu}(x)\delta g_{\mu\nu}(x)\sqrt{-g}\, d^4x$$

Read LHS:

$$\frac{d}{d\lambda} S_D[\psi, \nabla_{g+\lambda\delta g}\psi]\Big|_{\lambda=0}$$

- linear in $\delta g_{\mu\nu} = \delta g_{\nu\mu}$ (test function)
- defines $T_{\mu\nu} = T_{\nu\mu}$ (distr.)
- computation may require partial integration

About $T^{\mu\nu}{}_{;\nu} = 0$: expresses invariance of action under change of coordinates.

Let φ_t be the flow with generating vector field X $(= 0$ on ∂D, hence $\varphi_t(D) = D)$

Then

$$\int_{\varphi_{-t}(D)} \mathcal{L}({\varphi_t}^*\psi, \nabla_{{\varphi_t}^* g}{\varphi_t}^*\psi)\sqrt{-g_{{\varphi_t}^*}}\, d^4x$$

is independent of $t \longrightarrow \frac{d}{dt}(\dots)\mid_{t=0} = 0$

$$\underset{\text{metric}}{\delta g} = \frac{d}{dt}{\varphi_t}^* g\Big|_{t=0} = L_X g$$
$$(\delta g)_{\mu\nu} = X^\lambda g_{\mu\nu,\lambda} + g_{\lambda\nu}X^\lambda{}_{,\mu} + g_{\mu\lambda}X^\lambda{}_{,\nu}$$
$$X_{\mu;\nu} + X_{\nu;\mu}$$

both expressions are tensor fields, agree in normal coordinates, hence in any

Thus, by $\delta_\psi S = 0$

$$\frac{d}{dt}(\dots)\Big|_{t=0} = -\int_D \underbrace{\tfrac{1}{2}T^{\mu\nu}(X_{\mu;\nu} + X_{\nu;\mu})}_{=T^{\mu\nu}X_{\mu;\nu}=(T^{\mu\nu}X_\mu)_{;\nu}-T^{\mu\nu}{}_{;\nu}X_\mu} \sqrt{-g}\, d^4x$$

$$\int_D \overbrace{(T^{\mu\nu}X_\mu)_{;\nu}}^{W^\nu} \sqrt{-g}\, d^4x = \int_{\partial D} T^{\mu\nu}X_\mu\sqrt{-g}\, do_\nu \underset{\uparrow \atop X \text{ auf } \partial D = 0}{=} 0$$

$\longrightarrow T^{\mu\nu}{}_{;\nu} = 0$ in all of D

Full action:

$$\int_D (\frac{1}{2}R + \Lambda + \kappa\mathcal{L})\sqrt{-g}\, d^4x$$

$\longrightarrow \delta_g(\dots) = 0: \quad G_{\mu\nu} - \Lambda g_{\mu\nu} = \kappa T_{\mu\nu}$
(Note: Palatini method may not work)

Example: The freely falling e.m. field
Basic fields: e.m. potentials A_μ

Lagrangian:

$$\begin{aligned}\mathcal{L} &= +\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} \\ &= -\tfrac{1}{4}F_{\mu\nu}F_{\sigma\rho}g^{\sigma\mu}g^{\rho\nu}\end{aligned}$$

where $F_{\mu\nu} = A_{\nu;\mu} - A_{\mu;\nu} = A_{\nu,\mu} - A_{\mu,\nu}$

$$\frac{\partial\mathcal{L}}{\partial A^\nu} = 0, \qquad \frac{\partial\mathcal{L}}{\partial A_{\nu;\mu}} = -\frac{1}{4}F_{\sigma\rho}g^{\sigma\mu}g^{\rho\nu}\cdot 4 = -F^{\mu\nu}$$

E.-L. equations:

$$F^{\mu\nu}{}_{;\mu} = 0 \qquad \text{(Maxwell's eqs. in free fall)}$$

energy-momentum tensor

$$\delta_g \int_D \mathcal{L}\sqrt{-g}\, d^4x = \int_D \left[(\delta_g\mathcal{L}) + \frac{1}{2}\delta g^{\alpha\beta}\delta g_{\alpha\beta}\right]\sqrt{-g}\, d^4x$$

with

$$\begin{aligned}\delta_g\mathcal{L} &= -\tfrac{1}{4}\left[F_{\mu\nu}F_{\sigma\rho}\left(g^{\mu\sigma}(\delta g^{\nu\rho}) + (\delta g^{\mu\sigma})g^{\nu\rho}\right)\right] \\ &= \tfrac{1}{2}F_{\mu\nu}F_{\sigma\rho}g^{\mu\sigma}g^{\nu\alpha}g^{\rho\beta}\delta g_{\alpha\beta} \\ &= \tfrac{1}{2}F_{\mu}{}^{\alpha}F^{\mu\beta}\delta g_{\alpha\beta}\end{aligned}$$

$$T^{\alpha\beta} = -\mathcal{L}g^{\alpha\beta} - F_{\mu}{}^{\alpha}F^{\mu\beta} = F^{\alpha}{}_{\mu}F^{\mu\beta} - \frac{1}{4}(F_{\mu\nu}F^{\mu\nu})g^{\alpha\beta}$$

same as in electrodynamics

6 The homogeneous isotropic universe

Goal: find "highly symmetricßolution of the FE in presence of dust/ideal fluid, representing the universe (Friedmann 1922)

Idea: universe is spatially homogeneous & isotropic on large scales:

Evidence:

- matter: not homogeneous on small scales:
  distance between:

stars:	$\sim$	1	ps: 326 light years
galaxies	$\sim$	10^6	ps
clusters	$\sim$	10^7	ps
largest structure	$\sim$	10^8	ps

  beyond that: matter $\sim$ homogeneous & isotropic

- radiation: cosmic microwave background
  isotropic up to 10^{-5}

6.1 The Ansatz

Time Slices (in suitable coordinates) are 3-dimensional Manifolds of constant scalar curvature. Introduce them as submanifolds

$$M_0 \subset \Re^4 : k[(x^1)^2 + (x^2)^2 + (x^3)^2] + (x^4)^2 = {R_0}^2,$$

with $k = 0, \pm 1, R_0 > 0$; metric g_0 on M_0 induced by

$$g = (dx^1)^2 + (dx^2)^2 + (dx^3)^2 + (dx^4)^2 \quad \text{on } \Re^4$$

List them up:

k	M_0 (3-dim)	curvature	symmetry group $\mathcal{L}$
$+1$	sphere	> 0	$O(k)$
0	plane	$= 0$	$E(3)$: Euclidean motions
-1	hyperboloid	< 0	$L(k)$

M_0 is "highly symmetric": for $S \in \mathcal{S}$

- $S(M_0) = M_0$
- $S^* g_0 = g_0$
- any points $p, p' \in M_0$ are equivalent: $\exists S \in \mathcal{S} : p' = S(p)$ (homogenity)
- any two normalized vectors $V, V' \in T_{p_0}(M)$ are equivalent:
  $\exists S \in \mathcal{S} : S(p_0) = p_0$ and $V \leq S_* V$ (isotropy)

Fact: Any Riemannian Manifold of sign $(+++)$ and constant curvature is locally one of the above.

Charts:

A: coordinates (x^1, x^2, x^3, x^4)

$$x^4 = \sqrt{{R_0}^2 - kr^2} \equiv w(r), \qquad r = \sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2}$$

(for $k = +1$: upper hemisphere only)

$$\frac{\partial x^4}{\partial x^i} = \frac{1}{2w(x)}(-k)\frac{\partial x^2}{\partial x^i} = -\frac{kx^i}{w}$$

$$dx^4 = \sum_{i=1}^{3} \frac{\partial x^4}{\partial x^i} dx^i = -\frac{k}{w} \sum_{i=1}^{3} x^i dx^i$$

$$g_0 = \sum_{i=1}^{3} (dx^i)^2 + \frac{k}{{R_0}^2 - kr^2} \sum_{i,j=0}^{3} x^i x^j dx^i dx^j$$

B: coordinates (r, θ, φ)

$$x^1 = r\cos\theta\cos\varphi \qquad x^3 = r\sin\theta$$

$$x^2 = r\cos\theta\sin\varphi \qquad dx^4 = w'(r)dr = -\frac{kr}{w}dr$$

$$g_0 = r^2\left((d\theta)^2 + \sin^2\theta(d\varphi)^2\right) + \underbrace{(dr)^2 + \frac{kr^2}{w^2}(dr)^2}_{=\left(1+\frac{kr^2}{w^2}\right)dr^2 = \frac{R_0}{w}dr^2}$$

Variant: coordinates (χ, θ, φ)

$$r = R_0 \begin{cases} \sin\chi \\ \chi \\ \sinh\chi \end{cases} \qquad w(r) = R_0 \begin{cases} \cos\chi & k = +1 \\ 1 & k = 0 \\ \cosh\chi & k = -1 \end{cases}$$

$$g_0 = {R_0}^2 \left[d\chi^2 + \left\{ \begin{matrix} \sin^2\chi \\ \chi^2 \\ \sin^2\chi \end{matrix} \right\} \left(d\theta^2 + \sin^2\theta d\varphi^2\right) \right]$$

Spacetime: $M = I \times M_0$, $\quad I \subset \Re$ (interval)
Metric

$$\underbrace{g = dt^2 - a^2(t)g_0,}_{\text{Friedmann-Metric}} \qquad a(t) > 0 \qquad (c = 1)$$

(warped product)
The only 4-velocity field consistent with isotropy is

$$u = (1, 0, 0, 0)$$

Particles moving with u have constant coordinates in charts A,B: comoving coordinates. For such particles t is not only coordinate time but proper time.

Consequences:

- Hubble law: $d(t)$: spatial distance between any two such particles
  $d(t) = a(t)d_0$

  expansion rate: $\frac{\dot{d}(t)}{d(t)} = \frac{\dot{a}(t)}{a(t)} = H(t)$ $\quad \rightarrow$ Hubble constant

  is the same for all pairs: relative velocity is proportional to velocity:

  $\dot{d}(t) = H(t)d(t)$; Today: $H(\text{now}) \cong 72 \frac{\text{km/s}}{Mpc}$

- cosmological redshift ν_i: frequencies, $\quad \nu_i \Delta\tau^{(i)} = 1$

  $\frac{\nu_2}{\nu_1} = \frac{\Delta\tau^{(1)}}{\Delta\tau^{(2)}}$

  $(1), (2)$ at rest, comoving, $x(t) = (t, x(t))$

  $\vec{x}(t)$ runs radially, by isotropy

  $dt = a(t)\frac{R_0}{w}dr$, since light runs along null geodesic and $\vec{x}(t)$ radially

  hence

$$\int_0^r \frac{dr}{w} = {R_0}^{-1} \int_{t_1}^{t_2} \frac{dt}{a(t)} = {R_0}^{-1} \int_{t_1+\Delta t_1}^{t_2+\Delta t_2} \frac{dt}{a(t)}$$

$$\Rightarrow \frac{\Delta t_1}{a(t_1)} = \frac{\Delta t_2}{a(t_2)} \Rightarrow \frac{\nu_2}{\nu_1} = \frac{\Delta\tau^{(1)}}{\Delta\tau^{(2)}} = \frac{a(t_1)}{a(t_2)}, \text{ i.e.}$$

  If the universe is expanding, i.e. $a(t_2) > a(t_1)$ then $\nu_2 < \nu_1$,

  write $\frac{\nu_1}{\nu_2} = 1 + \underset{\substack{\uparrow \\ \text{redshift}}}{z}$ $\qquad$ Observations: z up to $7, 8$

Remark:
$(R_0, a(t))$ and $(\lambda R_0, \lambda^{-1}a(\lambda))$ describe the same model (redundancy)
Set $R_0 = 1$, formally replace $k/{R_0}^2 \rightsquigarrow k$

Ansatz: ideal fluid, $T^{\mu\nu} = (\rho + p)u^\mu u^\nu - pg^{\mu\nu}$ with $p = p(\rho)$ (equation of state) with $u^\mu = (1, 0, 0, 0)$ (by isotropy), $\rho = \rho(t)$ (by homogeneity)

6.2 The field equations

To show: FE satisfied by suitable choice of $a(t), \rho(t)$
By symmetry: Enough to consider a point $(t, 0, 0, 0)$ for all t.
Curvature contains $\partial^2 g \rightsquigarrow$ enough to keep Taylor expansion of $g(t, x^1, x^2, x^3)$ up to 2^{nd} order in $\vec{x}$. We have

$$g_{\mu\nu} = \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & \\ 0 & -a^2(\delta_{jk} + kx^jx^k) \\ 0 & \end{array}\right)$$

Hence

$$\begin{aligned} g_{\mu\nu,\sigma} &= 0, && \mu = 0 \text{ or } \nu = 0 \\ g_{ik,0} &= -2a\dot{a}\delta_{ik} && \text{1st order is enough} \\ g_{ik,l} &= -a^2 \end{aligned}$$

Remember: $\Gamma^\mu{}_{\nu\sigma} = \frac{1}{2} g^{\mu\rho}\left(g_{\nu\rho,\sigma} + g_{\sigma\rho,\nu} - g_{\nu\sigma\rho}\right)$

<u>Result:</u>

$$\begin{aligned} \Gamma^0{}_{ii} &= -\tfrac{1}{2}(-2a\dot{a}) = a\dot{a} \\ \Gamma^i{}_{i0} &= \Gamma^i{}_{0i} = \frac{\dot{a}}{a} \\ \Gamma^i{}_{ll} &= kx^i \\ &\text{others vanish} \end{aligned}$$

- Ricci tensor:

$$\begin{aligned} R_{00} &= -3\ddot{a}/a \\ R_{jj} &= a\ddot{a} + 2\dot{a}^2 + 2k \end{aligned}$$
$$(\text{others } = 0; \quad R_{jk} \propto \delta_{jk} \text{ by isotropy})$$

e.g.

$$\begin{aligned} R_{00} = R^\alpha{}_{0\alpha0} &= \underbrace{\Gamma^\alpha{}_{00,\alpha}}_{=0} - \underbrace{\Gamma^\alpha{}_{\alpha00}}_{-3\left(\frac{\dot{a}}{a}\right)^\bullet} + \underbrace{\Gamma^\sigma{}_{00}}_{=0}\Gamma^\alpha{}_{\alpha\sigma} - \underbrace{\Gamma^\sigma{}_{\alpha0}\Gamma^\alpha{}_{0\sigma}}_{\substack{\sigma\to i \\ \alpha\to i \\ -3\left(\frac{\dot{a}}{a}\right)^2}} \\ &= -3\left(\frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2} + \frac{\dot{a}^2}{a^2}\right) \end{aligned}$$

- scalar curvature:

$$\begin{aligned} R = R^\mu{}_\mu &= R_{00} - \frac{1}{a^2}\sum_{i=1}^{3} R_{ii} \\ &= -\frac{3}{a^2}\left(a\ddot{a} + a\ddot{a} + 2\dot{a}^2 + 2k\right) = -\frac{6}{a^2}\left(a\ddot{a} + \dot{a}^2 + k\right) \end{aligned}$$

- Einstein tensor:

$$G_{\mu\nu} = R_{\mu\nu} - \tfrac{1}{2}Rg_{\mu\nu}, \qquad \text{is diagonal}$$
$$G_{00} = \frac{3}{a^2}\left(\dot{a}^2 +\right)$$
$$G_{jj} = -\left(2a\ddot{a} + \dot{a}^2 + k\right)$$

- Energy-momentum tensor:

$$T_{00} = \rho$$
$$T_{jj} = pa^4$$

- FE: (only interesting parts are diagonal ones)

$$G_{\mu\nu} - \Lambda g_{\mu\nu} = \kappa T_{\mu\nu}$$

$$\left.\begin{aligned} &\text{(00)-component:} \quad \frac{3}{a^2}\left(\dot{a}^2 + k\right) - \Lambda = \rho \\ &\text{(jj)-component:} \quad +(2a\ddot{a} + \dot{a}^2 + k - \Lambda a^2 = -pa^2 \end{aligned}\right\} \quad \textbf{Friedmann Equations}$$

Remarks:

1. $a(t), \rho(t)$ are solutions. Then so are $a(t-t_0), \rho(t-t_0)$ and $a(-t), \rho(-t)$
2. "1st law of thermodynamics"

$$\frac{d}{dt}\left(\tfrac{1}{3}\rho a^3\right) = \dot{a}(\dot{a}^2 + k) + a2\dot{a}\ddot{a} - \Lambda a^2\dot{a} = \dot{a}(2a\ddot{a} + \dot{a}^2 + k - \Lambda a)$$
$$= -pa^2\dot{a} = -p\frac{d}{dt}(\underbrace{\tfrac{1}{3}a^3}_{\text{Volume}})$$

replaces 2nd Friedmann equation

3. 1st law is

$$0 = T^{\mu\nu}{}_{;\nu} = T^{\mu\nu}{}_{,\nu} + \Gamma^{\nu}{}_{\nu\rho}T^{\mu\rho} + \Gamma^{\mu}{}_{\nu\rho}T^{\rho\nu}$$

for $\mu = 0$:

$$T^{0\nu}{}_{;\nu} = \dot{\rho} + 3\frac{\dot{a}}{a}\rho + 3a\dot{a}\frac{p}{a^2}$$
$$= \frac{1}{a^3}\left[\frac{d}{dt}\left(\rho a^3\right) + p\frac{d}{dt}a^3\right]$$
$$\overset{!}{=} 0$$

4. Equation of state $p = W\rho$

   $W = 0$: dust, $W = \frac{1}{3}$: isotropic e.m. radiation, ($W = -1$: Vacuum)

$$\frac{d}{dt}(\rho a^3) \cdot a^{3W} = -W\rho \frac{d}{dt} a^3 \cdot a^{3W}$$

$$\Rightarrow \frac{d}{dt}\left(\rho a^{3(1+W)}\right) = 0$$

$$\rho \propto a^{-3(1+W)} = \begin{cases} a^{-3} & W = 0 \\ a^{-4} & W = \frac{1}{3} \\ a^{0} & W = -1 \end{cases}$$

   Universe goes from being radiation dominated to matter to vacuum dominated

   Henceforth: dust (ΛCDM) Λ cold dark matter

$$\frac{1}{3}\rho a^3 = C > 0 \qquad \text{constant}$$

   Then

$$\underbrace{\dot{a}^2 - \frac{1}{3}\Lambda a^2 - \frac{C}{a}}_{V(a)} = -k$$

   Analogy: Energy conservation of particles moving in 1-dim. $\left(\frac{1}{2}m = 1\right)$ in a potential $V(a)$ and total energy $-k$

   Special cases:

   (a) static universe (Einstein 1917)

$$k = +1: \quad V(a) = -1, \qquad V'(a) = 0$$

$$\downarrow \qquad\qquad \downarrow$$

$$-\Lambda a^2 = -1 \quad \leftarrow \quad -\frac{2}{3}\Lambda a + \frac{C}{a^2} = 0 \Rightarrow C = \frac{2}{3}a^3\Lambda$$

   Hence

$$a = \Lambda^{-\frac{1}{2}}, \quad \rho = 2\Lambda \qquad \text{unstable equilibrium}$$

   (b) de Sitter universe (1917)

   $C = 0$ no matter, $\Lambda = 0$

$$\dot{a}^2 - \frac{1}{3}\Lambda a^2 = -k$$

$$a(t) = \begin{cases} \frac{1}{\alpha}\cosh \alpha t & k = +1 \\ \frac{1}{\alpha} e^{\alpha t} & k = 0 \\ \frac{1}{\alpha}\sinh \alpha t & k = -1 \end{cases}$$

   with $\alpha^2 = \frac{\Lambda}{3}$

(c) $\Lambda = 0:$ $V(a) = -\frac{C}{a}$ Big Bang or Big Crunch (or both if $k = +1$)

Solutions $a(t)$ with $a(0) = 0$

Parametric representation

$k = +1$:

$$a = \tfrac{1}{2}C(1 - \cos\eta)$$
$$t = \tfrac{1}{2}C(\eta - \sin\eta) \qquad (0 < \eta < 2\pi)$$

$k = 0$:

$$a = \left(\frac{9C}{k}\right)^{\frac{1}{3}} t^{\frac{2}{3}} \qquad \text{Einstein-de Sitter universe}$$

$k = -1$:

$$a = \tfrac{1}{2}C(\cosh\eta - 1)$$
$$t = \tfrac{1}{2}C(\sinh\eta - \eta) \qquad (0 < \eta < 2\infty)$$

e.g. $k = +1$:

$$da = \tfrac{1}{2}C\sin\eta d\eta$$
$$dt = \tfrac{1}{2}C(1 - \cos\eta)d\eta$$

$$\dot{a} = \frac{da}{dt} = \frac{\sin\eta}{1 - \cos\eta} \Longrightarrow \dot{a}^2 = \frac{\sin^2\eta}{(1 - \cos\eta)^2} = \frac{1 + \cos\eta}{1 - \cos\eta}$$
$$-\frac{C}{a} = -\frac{\eta}{1 - \cos\eta}$$

$$\Rightarrow \dot{a}^2 - \frac{C}{a} = \frac{\cos\eta - 1}{1 - \cos\eta} = -1$$

General case: solutions are parametrized by

$$\Lambda, C, a(t_0)$$

Instead (usual case in cosmology): - t_0 today
- new parameters (reflecting today's properties of universe)

Reintroduce R_0:

$$\dot{a}^2 - \frac{1}{3}\Lambda a^2 - \frac{C}{a} = -\frac{k}{{R_0}^2}$$

Divide by $\dot{a}(t_0)^2$ ($\neq 0$, excludes static solutions)

$$\left(\frac{\dot{a}(t)}{\dot{a}(t_0)}\right)^2 - \frac{1}{3}\Lambda\left(\frac{a(t)}{\dot{a}(t_0)}\right)^2 - \frac{1}{3}\frac{\rho(t_0)a(t_0)^3}{\dot{a}(t_0)^2 a(t)} = -\frac{k}{{R_0}^2\dot{a}(t_0)}$$

Pick R_0 so that $a(t_0) = 1$ (g_0 describes distances today)

$$H \equiv H(t_0) = \frac{\dot{a}(t_0)}{a(t_0)} = \dot{a}(t_0)$$

$$\frac{\dot{a}^2}{H^2} - (\Omega_\Lambda a^2 + \Omega_m a^{-1}) = -\frac{k}{{R_0}^2 H^2} =: \Omega_k \underset{\overset{\uparrow}{t=t_0}}{=} 1 - \Omega_\Lambda - \Omega_m \qquad \bigstar$$

with

$$\Omega_\Lambda = \frac{1}{3}\frac{\Lambda}{H^2}, \qquad \Omega_m = \frac{\rho(t_0)}{3H^2}$$

New parameters: $H, \Omega_\Lambda, \Omega_m$ determine also
$k = -sign\,\Omega_k$

$\bigstar$ is energy conservation of non-relativistic particle (mass $\frac{2}{H^2}$), potential

$$U(a) = -(\Omega_\Lambda a^2 + \Omega_m a^{-1})$$

total energy Ω_k
Depending on $sign\,\Omega_\Lambda = sign\,\Lambda$ we get different types of motion:

$\underline{\Lambda = 0}$ $\Omega_k = 1 - \Omega_m$

$\Omega_m < 1$ indefinite expansion a(t) with

$$\lim_{t\to\infty} \dot{a}(t) > 0$$

$\Omega_m = 1$ indefinite expansion

$$\dot{a}(t) \overset{t\to\infty}{\longrightarrow}$$

$\Omega_m > 1$ finite expansion, recollapse

$\underline{\Lambda < 0}$ finite expansion, recollapse

$\underline{\Lambda > 0}$ U(a) has maximum

$$U(a_{max}) = -3\Omega_\Lambda^{\frac{1}{3}}(\frac{\Omega_m}{2})^{\frac{2}{3}}$$
$$a_{max} = (\frac{\Omega_m}{2\Omega_\Lambda})^{\frac{1}{3}}$$

If $a_{max} > a(t_0) = 1$, i.e.

$$\Omega_m > 2\Omega_\Lambda$$

then expansion is deceleration

Motion is bounded (from above or below) if

$$1 - \Omega_\Lambda - \Omega_m < -3\Omega_\Lambda{}^{\frac{1}{3}}(\frac{\Omega_m}{2})^{\frac{2}{3}}$$

can occur only for $\Omega_\Lambda + \Omega_m > 1$

-if Ω_Λ small: $1 - \Omega_m < -3\Omega_\Lambda{}^{\frac{1}{3}}(\frac{\Omega_m}{2})^{\frac{2}{3}}$

$\frac{\Omega_\Lambda}{\Omega_m} < 4\left(\frac{\Omega_m - 1}{3\Omega_m}\right)^3$

-if Ω_m small: $\frac{\Omega_m}{\Omega_\Lambda} < 2\left(\frac{\Omega_\Lambda - 1}{3\Omega_\Lambda}\right)^{\frac{2}{3}}$

Age of universe:

- in decelerating models ($\ddot{a}(t_0) \leq 0$): $\ddot{a}(t) \leq 0$ in the past $t \leq t_0$
  Thus $t_0 \leq H^{-1}$ (Hubble time)
- In general:

$$\frac{\dot{a}^2}{H^2} = \Omega_k - U(a)$$
$$\frac{da}{dt} = H\sqrt{\Omega_k - U(a)}$$
$$t_0 = \int_0^{t_0} dt = H^{-1}\int_0^1 \frac{da}{\sqrt{\Omega_k - U(a)}}$$

6.3 Which universe do we live in?

Observations $\longrightarrow H, \Omega_\Lambda, \Omega_m$

$H = \frac{\dot{a}(t_0)}{a(t_0)}$ from redshift & distance: light from far away galaxies

t_s: sending time, t_0: receiving time

$$a(t_s) = a(t_0) - \dot{a}(t_0)(t_0 - t_s) + \frac{1}{2}\ddot{a}(t_0)(t_0 - t_s)^2 + \dots$$
$$= a(t_0)\left[1 - H(t_0 - t_s) - \frac{1}{2}H^2 q(t_0 - t_s)^2 + \dots\right]$$

for $t_0 - t_s$ small compared to age of universe
$q = -\frac{a(t_0)\ddot{a}(t_0)}{\dot{a}(t_0)^2}$ "deceleration parameter"
$\frac{v_s}{v_0} = 1 + z = \frac{a(t_0)}{a(t_s)}$
$\Rightarrow$ $1 + z = 1 + H(t_0 - t_s) + H^2(1 + \frac{1}{2}q)(t_0 - t_s)^2 + \dots$

Distance today:

$$d = a(t_0)R_0 \int_0^r \frac{dr'}{w(r')} = a(t_0) \int_{t_s}^{t_0} \frac{dt}{a(t)}$$
$$= (t_0 - t_s) + \frac{1}{2}H(t_0 - t_s)^2$$

Eliminate $t_0 - t_s$ from equations

$$z = Hd + \frac{1}{2}(1 + q)(Hd)^2 + \dots \qquad \text{(distance-redshift relation)}$$

Lowest order: interpret as Doppler: $1 + z = 1 + \frac{v}{c}$

$$z = \dot{d}(t_0) = H(t_0)d(t_0)$$

$H = \frac{z}{d}$:

- z from spectra (emission or absorption)
- d standard candles (Cepheids, Supernovae of type Ia)

In higher order $\longrightarrow q$

$$2q = \Omega_m - 2\Omega_\Lambda \qquad \text{from} \qquad \frac{\dot{a}^2}{H^2} - (\Omega_\Lambda a^2 + \Omega_m a^{-1}) = \Omega_k$$

Cosmic Microwave Background (CMB) = black body radiation at T = 2.73 K isotropic

Origin: Nuclei & electrons combined to neutral atoms at T = 3000 K
Neutral atoms are transparent to e.m. radiation

$\Rightarrow$ red-shifted ever since by $1 + z = \frac{3000 \text{ K}}{2.71 \text{ K}} = 1100$

$$a(t_s) = \frac{a(t_0)}{1 + z} = \frac{1}{z}$$

1st Friedmann

$$\frac{\dot{a}(t_s)^2}{H^2} - (\underbrace{\Omega_\Lambda a(t_s)^2}_{=0} + \Omega_m a(t_s)^{-1}) = 1 - \underbrace{\Omega_\Lambda}_{=0} - \Omega_m$$
$$H(t_s)^2 = \frac{\dot{a}(t_s)^2}{a(t_s)^2} = H^2\Omega_m a(t_s)^{-3} = H^2\Omega_m z^3$$

Intensity fluctuations of order 10^{-5} of CMB
correlation length (“standard rules“)

$$\Delta s = 2H(t_s)^{-1} \qquad (\dots)$$

Seen on ... at

$$\Delta\varphi \approx 1^\circ$$

$z, \Delta s, \Delta\varphi$ determine geometry: open, flat, closed

$$\Delta s = a(t_s) r \Delta\varphi = z^{-1} r \Delta\varphi$$
$$\Delta s = 2H(t_s)^{-1} = 2H^{-1}{\Omega_m}^{-\frac{1}{2}} z^{-\frac{3}{2}}$$
$$\frac{r}{R_0} = 2\left(\frac{\Omega_k}{\Omega_m}\right)^{\frac{1}{2}} z^{-\frac{1}{2}} (\Delta\varphi)^{-1}$$
$$\frac{r}{R_0} = \left\{\begin{matrix} \sin\chi & k=+1 \\ \chi & k=0 \\ \sinh\chi & k=-1 \end{matrix}\right\} := \operatorname{sinn}\chi$$

with $d\chi = \frac{dr}{dw(r)}$
$$w(r) = \sqrt{{R_0}^2 - kr^2}$$
$$\chi = \int_0^1 \frac{dr}{w(r)} = {R_0}^{-1} \int_{t_s}^{t_r} \frac{dt}{a(t)} = {R_0}^{-1} \int_0^1 \frac{da}{a\dot{a}} = (\Omega_k)^{\frac{1}{2}} \int \frac{da}{a\sqrt{\Omega_k - U(a)}}$$

Another constraint on Ω_Λ, Ω_m

$$\Omega_\Lambda + \Omega_m = 1.02 \pm 0.02$$

Altogether:

$$\Omega_m = 0.27 \pm 0.04$$
$$\Omega_\Lambda = 0.73 \pm 0.04$$

(baryonic dark matter)
→CMB power spectrum→0.02−0.04

Age of universe:

$$\approx 1H^{-1} = 13.7 \cdot 10^9 \text{ years}$$

6.4 The causality and the flatness problems

Conformal time η: $dt = R_0 a(t) d\eta$
Thus:

$$g = {R_0}^2 a(t)^2 \left(d\eta^2 - (d\chi^2 + \operatorname{sinn}^2\chi((d\theta)^2 + \sin^2\theta d\varphi^2))\right)$$

Normalize $\eta = 0$ at $t = 0$

$$\eta = {R_0}^{-1} \int_0^1 \frac{dt'}{a(t')} \qquad \star\star$$

possible if integral is convergent at $t' = 0$.

Equation of state $p = W\rho$ $(W = const.)$

$$\rightarrow a(r) \propto t^\alpha, (t \rightarrow 0), \alpha = \frac{2}{3+W}$$

★★ is convergent if $\alpha < 1$, i.e. $W > -1$

Then, for $t \rightarrow 0$

$$\eta \propto \frac{t}{a(t)} \propto \frac{1}{\dot{a}(t)}$$

more precisely

$$\eta(t) \approx \frac{2}{1+W} \left(R_0 \dot{a}(t)\right)^{-1}$$

Geodesics ending at $\chi = 0$ come in radially ($d\theta = d\varphi = 0$)

$$g = {R_0}^2 a(t)^2 \left(d\eta^2 - d\chi^2\right) :$$

conformally equivalent to $(1+1)$-Minkowski
$\rightarrow$ null geodesics run at $\pm 45°$

Observers at χ (fixed) causally connected to p only for $\chi \leq \eta$
i.e. at most at distance

$$d = R_0 a(t) \eta = \frac{2}{1+W} \frac{a(t)}{\dot{a}(t)} = \frac{2}{1+W} H(t)^{-1}$$

For $t = t_s$ and $W = 0$:

$$d = 2H(t_s)^{-1} \qquad \text{(today: at } \Delta\varphi \approx 1°$$

)
CMB is homogeneous on all of sky, thus includes regions causally disconnected at t_s

CAUSALITY PROBLEM!

Possible solution: inflation ($W \approx 1$)
$\rightarrow \eta$ is unbounded below
$\Rightarrow$ causality problem disappears

FLATNESS PROBLEM:

$$1 - \Omega_\Lambda - \Omega_m = -\frac{k}{{R_0}^2 \dot{a}(t_0)^2} = \Omega_k$$

Similarly in the past:

$$\Omega_k(t) = -\frac{k}{{R_0}^2 \dot{a}(t_0)^2}$$

$$\frac{\Omega_k(t)}{\Omega_k} = \frac{\dot{a}(t_0)^2}{\dot{a}(t)^2} = \frac{H^2}{\dot{a}(t)^2} \qquad a(t_0) = 1$$

$$= \frac{1}{\Omega_k + \Omega_\Lambda a^2 + \Omega_m a^{-1}}$$

$\Rightarrow$ In the past, $a(t) \rightarrow 0$

$$\Omega_k(t) \rightarrow 0$$

Universe must have been much flatter in the past (so that it is still quite flat today):

Possible solution: inflation
because it drives $\frac{\Omega_k(t)}{\Omega_k} \rightarrow 0$

7 The Schwarzschild-Kruskal metric

7.1 Stationary and static metrics

(M, g) pseudo-Riemannian manifold. Let $\varphi_s : M \to M$ with ${\varphi_s}^* g = g$ be a flow of isometrics

Generation vector fields of φ_s:

$$Kf = \frac{d}{ds} \left(f \circ \varphi_s \right) \Big|_{s=0}$$

satisfies

$$L_K g = \frac{d}{ds} \left({\varphi_s}^* g \right) \Big|_{s=0} = 0$$

Definition: A vector field K with $L_K g = 0$ is a **Killing Field**

Definition: A vector field V with $g(V, V) = \begin{cases} > 0 & \text{is timelike} \\ = 0 & \text{is lightlike} \\ < 0 & \text{is spacelike} \end{cases}$

Definition: A metric g is (locally) stationary if there is a chart so that

$$g_{\mu\nu,0} = 0 \quad \text{and} \quad \tfrac{\partial}{\partial x^0} \text{ timelike}$$

Then $K = \frac{\partial}{\partial x^0}$, i.e. $K^\mu = (1, 0, 0, 0)$ and

$$(L_K g)_{\mu\nu} = \underbrace{K^\lambda g_{\mu\nu,\lambda}}_{g_{\mu\nu,0}} + g_{\lambda\nu} \underbrace{K^\lambda{}_{,\mu}}_{=0} + g_{\mu\lambda} \underbrace{K^\lambda{}_{,\nu}}_{=0}$$

So K is a timelike Killing field. Conversely, g is stationary if there is a timelike so that $L_K g = 0, (K, K) = 0$.

Proof: By construction of a chart where ... holds true. Let φ_t be the flow generated by K and $M \supset F$ be a spacelike 3-surface. (i.e. its tangent vectors are spacelike) with some coordinates $(x^1, x^2, x^3) \leftrightarrow p_0$
Set $(t, x^1, x^2, x^3) \leftrightarrow \varphi_t(p_0) \in M$. In this chart

$\varphi_s(t, x^1, x^2, x^3) = (t + s, x^1, x^2, x^3)$ and

$K^\mu = (1, 0, 0, 0)$

So

$0 = (L_K g)_{\mu\nu} = K^\lambda g_{\mu\nu,\lambda} + 0 + 0 = g_{\mu\nu,0}$

Definition: A metric is (locally) static if in a chart $(x^\mu = (x^0, \vec{x}))$

$$\underbrace{\frac{\partial}{\partial x^0}\text{ timelike}}_{\Leftrightarrow g_{00}(\vec{x})\geq 0} \quad \text{and} \qquad g = g_{\mu\nu}dx^\mu dx^\nu = g_{00}(\vec{x})(dx^0)^2 + \sum_{i,k=1}^{3} g_{ik}(\vec{x})dx^i dx^k$$

I.e. static $\equiv$ stationary & $g_{0j} = 0$

Remark: In a (slowly) rotating frame, $g_{0i} = -\frac{1}{c}(\vec{\omega} \wedge \vec{x})_i$
So when g is static there is a globally non-rotating field (and vice-versa)

Intrinsic formulation: $K = \frac{\partial}{\partial x^0}, K^\mu = (1,0,0,0)$ is timelike Killing Field

$$\underset{\substack{\uparrow \\ \text{1-form}}}{\hat{K}} = gK \quad \rightarrow \quad \hat{K}_\mu = (g_{00}, 0, 0, 0)$$

$$\hat{K} = \hat{K}_\mu dx^\mu = g_{00}dx^0, d\hat{K} = dg_{00} \wedge dx^0$$

$$\rightarrow \quad \hat{K} \wedge d\hat{K} = dx^0 \wedge (dg_{00} \wedge dx^0) = 0$$

7.2 The Schwarzschild metric

Ansatz: for metric $g = ds^2$ solving the FE in vacuum

$$R^{\mu\nu} = \kappa_0 \left(T^{\mu\nu} - \frac{1}{2}Tg^{\mu\nu}\right) = 0$$

shall describe exterior of spherically symmetric non-rotating star
(↑ because there is vacuum)

(in classical physics: $\varphi = -\frac{G_0 M}{r}$)

$$ds^2 = e^{2a(r)}dt^2 - \left[e^{2b(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2)\right] \qquad ⋇$$

on $M = \underset{\substack{\in \\ t_0}}{\Re} \times \underset{\substack{\in \\ r}}{I} \times \underset{\substack{\in \\ \vec{e} \\ \Leftrightarrow \\ (\theta,\varphi)}}{S^2}$, with $a(r), b(r)$ arbitrary (later to be determined from $R_{\mu\nu} = 0$ (FE))

Remarks:

1. Metric is static, invariant under rotations of S^2
2. The most general which is static, spherically symmetric is of the form ⋇ in suitable coordinates (without proof)

   Definition: (M, g) spherically symmetric:

   (a) $SO(3) \ni R$ acts on M as isometrics, i.e. $R: M \rightarrow M, p \mapsto R(p)$ with $R^*g = g$

   (b) for each $p \in M$, the "orbit" $\{R(p) \in M | R \in SO(3)\}$ is spacelike 2-surface
3. Replacing $r^2 \rightarrow f(r)$ $\quad (f > 0$ arb.) still satisfies property (a)

Keeping r^2: Area of the sphere with coordinate r is $4\pi r^2$
Length of great circle on there is $2\pi r$
(but radius $\neq r$)

4. Transition function $\tilde{t} \underset{(r,\theta,\varphi\text{: fixed})}{\mapsto} t = e^c \tilde{t} \quad (\to dt = e^c d\tilde{t})$

   amounts to

$$a(r) \rightsquigarrow a(r) + c = \tilde{a}(r)$$

   $\to a, \tilde{a}$ represents the same spacetime (metric) but in different charts

Christoffel symbols: check selected symbols

$$\Gamma^t{}_{tr} = \tfrac{1}{2} g^{\overset{\text{summation index, the only one giving result} \neq 0 \text{ is t}}{t}\,t} (g_{tt,r} + \underbrace{g_{rt,t} - g_{tr,t}}_{\substack{\text{both } =0,\\ \text{because off-diagonal}}})$$

$$= \tfrac{1}{2} e^{-2a(r)} \frac{d}{dr}\left(e^{2a(r)}\right) = \frac{1}{2} 2a'(r) = a'(r)$$

$$\Gamma^r{}_{tt} = \tfrac{1}{2} g^{rr} (\underbrace{g_{tr,t}}_{=0} + \underbrace{g_{tr,t}}_{=0} - g_{tt,r})$$

$$= \tfrac{1}{2}\left(-e^{-2b(r)}\right) \frac{d}{dr}\left(-e^{2a(r)}\right) = a' e^{2(a-b)}$$

$$\Gamma^t{}_{tt} = \tfrac{1}{2} g^{tt} (\underbrace{g_{tt,t}}_{=0\text{, metric stationary}} + g_{tt,t} - g_{tt,t}) = 0$$

Ricci:

$$R_{tt} = \underset{\alpha=r}{\Gamma^\alpha{}_{tt,\alpha}} - \underbrace{\Gamma^\alpha{}_{\alpha t,t}}_{=0} + \underset{\sigma=r}{\Gamma^\sigma{}_{tt}} \Gamma^\alpha{}_{\alpha\sigma} - \underbrace{\Gamma^\sigma{}_{\alpha t} \Gamma^\alpha{}_{t\sigma}}_{\substack{\alpha = t, \quad \sigma = r\\ \alpha = r, \quad \sigma = t}}$$

$$= \frac{d}{dr} a' e^{2(a-b)} + a' e^{2(a-b)} \left(a' + b' + r^{-1} + r^{-1}\right) - a' e^{2(a-b)} a' \cdot 2$$

$$= \left(a'' - a'b' + a'^2 + 2\frac{a'}{r}\right) e^{2(a-b)}$$

Field equations in vacuum: $R_{\mu\nu} = 0$

- From $R_{tt} e^{-2(a-b)} + R_{rr} = 0$

$$\cancel{\frac{2}{r}}(a' + b') = 0$$

  hence $a + b = C = 0$ (without loss of generality by Remark 4)

- From $R_{00} = R_{\varphi\varphi} = 0$:

$$1 = e^{-2b} - 2rb'e^{-2b} = \left(re^{-2b}\right)'$$

$$\to re^{-2b} = r - 2m \qquad \text{(integration constant } m) \qquad \to e^{-2b} = 1 - \frac{2m}{r}$$

- From $R_{tt} = 0$:

$$-\left(-b'^2 + b'' - b'^2\right) - \frac{2b'}{r} = 0$$
$$\left(r\left(2b'^2 - b''\right) - 2b'\right)e^{-2b} = 0$$

already satisfied

$$\frac{d}{dr}\left(1 - 2rb'\right)e^{-2b} = \frac{d}{dr}1 = 0$$

<u>Result:</u>

$$g = ds^2 = \left(1 - \frac{2m}{r}\right)dt^2 - \left[\left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right)\right]$$

for $r \to \infty$: g tends to Minkowski metric in spherical coordinates

Newtonian potential φ in weak field

$$\varphi = \frac{c^2}{2}(g_{00} - 1) = -\frac{mc^2}{r} = -\frac{G_0 M}{r}$$
$$\Rightarrow m = \frac{G_0 M}{c^2} \quad (> 0)$$

At $r = 2m$ (Schwarzschild radius), $g_{\alpha\beta}$ becomes singular ($r > 2m$ for now) in the chart

- light cones: $ds^2 = 0$:

$$\left(\frac{dt}{dr}\right)^2 = \left(1 - \frac{2m}{r}\right)^{-2}$$
$$\left|\frac{dt}{dr}\right| = \pm\left|1 - \frac{2m}{r}\right|^{-1}$$

  degenerate at $r = 2m$

  infalling light, starting from (t_0, r_0)

$$dt = \left(1 - \frac{2m}{r}\right)(-dr)^{-1} = -\frac{r}{r - 2m}dr$$
$$\int_{t_0}^{t} dt = \int_{r}^{r_0} \frac{r}{r - 2m}dr$$

  $\to +\infty$ as $r \searrow 2m$

- The line $r = 2m$ (θ, φ fixed) is a <u>single</u> event: $d\tau^2 = \left(1 - \frac{2m}{r}\right)dt^2 = 0$

$$\left.\frac{d\tau}{dt}\right|_{\text{r fixed}} = \left(1 - \frac{2m}{r}\right)^{\frac{1}{2}} \underset{r \searrow 2m}{\to} 0$$

- One finds: $(R=0)$

$$R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} = \frac{48m^2}{r^2} \qquad \text{(invariant)}$$

regular at $r = 2m$

- We'll see: there is a chart extending past $r = 2m$

Example: Sun: $r = 2m \cong \underset{\substack{\uparrow \\ \text{irrelevant}}}{3}$ km, $R_0 = 7 \cdot 10^5$ km

7.3 Geodesics in the Schwarzschild metric

- timelike geodesics: free fall of a body
  $\longrightarrow$ orbits of planets
  $\longrightarrow$ deviations from Kepler's law
  (perihelion advance)[1]

- null geodesics: light ray
  $\longrightarrow$ light deflection[1]

Lagrangian Function:

$$\mathcal{L} = g(\dot{x}, \dot{x}) \qquad \dot{} = \frac{d}{d\tau} \qquad \tau: \text{ affine parameter}$$

$$= \left(1 - \tfrac{2m}{r}\right)\dot{t}^2 - \left(1 - \tfrac{2m}{r}\right)^{-1}\dot{r}^2 - r^2\left(\dot{\theta}^2 + \sin^2\theta\,\dot{\varphi}^2\right)$$

(timelike: $\mathcal{L} = 1$, null geodesic $\mathcal{L} = 0$)

$\rightarrow\tau$: proper time

Geodesic equation:=Euler-Lagrange equations for $\mathcal{L}$

θ-equation:

$$\frac{\partial\mathcal{L}}{\partial\theta} = -2r^2\sin\theta\cos\theta\dot{\varphi}^2$$

$$\frac{\partial\mathcal{L}}{\partial\dot{\theta}} = -2r\dot{\theta}$$

$$\longrightarrow -(r^2\dot{\theta})^\bullet + r^2\sin\theta\cos\theta\dot{\varphi}^2 = 0$$

$\theta(t) \equiv \frac{\pi}{2}$ is solution: orbit is in equatorial plane

In general: initial values $\vec{e}, \dot{\vec{e}}$: $\vec{e} \perp \dot{\vec{e}}$
define plane in $\mathbb{R}^3$:
take it as equatorial plane
$\rightarrow \theta = \frac{\pi}{2}, \dot{\theta} = 0$
$\rightarrow \theta(t) \equiv \frac{\pi}{2}$
$\rightarrow$ orbits are planar

Alternatively: deduce from rotational symmetry by Noether

Lagrangian: (planar problem)

$$\begin{aligned}\mathcal{L} &= \mathcal{L}(\not{t}, r, \not{\varphi}, \dot{t}, \dot{r}, \dot{\varphi}, \not{\tau}) \\ &= \left(1 - \tfrac{2m}{r}\right)\dot{t}^2 - \left(1 - \tfrac{2m}{r}\right)^{-1}\dot{r}^2 - (r\dot{\varphi})^2\end{aligned}$$

φ, t cyclic variables: conservation law

$$\frac{1}{2}\frac{\partial \mathcal{L}}{\partial \dot{\varphi}} = -r^2\dot{\varphi} = -l \qquad (l: \text{ angular momentum})$$

$$\frac{1}{2}\frac{\partial \mathcal{L}}{\partial \dot{t}} = \left(1 - \frac{2m}{r}\right)\dot{t} = \epsilon \qquad \text{“energy“}$$

independence of τ: $\mathcal{L}$ conserved

Problem reduces to radial one:

$$\mathcal{L} = \left(1 - \frac{2m}{r}\right)^{-1}\left(\epsilon^2 - \dot{r}^2\right) - \frac{l^2}{r^2}$$

$$\dot{r}^2 + \underbrace{\left(1 - \frac{2m}{r}\right)\left(\mathcal{L} + \frac{l^2}{r^2}\right)}_{V(r)} = \epsilon^2$$

<u>Radial motion:</u>

- particle of mass 1 $\qquad (\mathcal{L} = 1)$

  $\frac{1}{2}\dot{r}^2 + \underbrace{\frac{1}{2}V(r)}_{\substack{\text{effective potential}\\ \text{for radial motion}}} = \frac{\epsilon^2}{2}$

  $$\frac{1}{2}V(r) = \underbrace{\underset{\substack{\uparrow\\ \text{additive}\\ \text{const.}}}{\frac{1}{2}} \; \underbrace{-\frac{m}{r}}_{\substack{\text{Newtonian}\\ \text{potential}}} \; \underbrace{+\frac{l^2}{2r^2}}_{\substack{\text{centrifugal}\\ \text{barrier}}}}_{\text{Non-relativistic}} \; \underbrace{-\frac{ml^2}{r^3}}_{\text{GR correction}} \qquad (m = G_0 M)$$

  Features:

  1. $l = 0$: $\quad \frac{1}{2}V(r) = \frac{1}{2} - \frac{m}{r}$ $\quad$ as with Newton
     particle crosses $r = 2m$ in <u>finite</u> proper time
  2. Even for $l > 0$: $\quad$ capture is possible
  3. $\frac{1}{2}V'(r) = \frac{m}{r^2} - \frac{l^2}{r^3} + \frac{3ml^2}{r^4} = \frac{1}{r^4}\left(mr^2 - l^2 r + 3ml^2\right)$
     l fixed:
     Non-relativistic: One circular orbit

     $$r_0 = \frac{l^2}{m}$$

GR: Either two or none:

$$r_\pm = \frac{l^2 \pm (l^4 - 12m^2 l^2)^{\frac{1}{2}}}{2m}$$

for $l^2 > 12m^2$ (none otherwise)
r_+ stable, r_- unstable

- light $(\mathcal{L}) = 0$
  $V(r) = \left(1 - \frac{2m}{r}\right)\frac{l^2}{r^2}$
  For $\epsilon^2 > \frac{l^2}{27m^2}$: capture **
  Meaning of l, ϵ: equation of straight line
  in polar coordinates:

$$r \sin\varphi = b$$
$$\dot r \underbrace{r \sin\varphi}_{0} + \underbrace{r^2 \dot\varphi}_{l} \cos\varphi = 0$$

at $t \mapsto \infty \quad (\varphi \to 0):$ $\qquad \dot r = \frac{dr}{d\tau} = \frac{dr}{dt}\frac{dt}{d\tau} \to -\epsilon$

$$-\epsilon b + l = 0$$
$$\Rightarrow \qquad b = \frac{l}{\epsilon}$$
$$** : \qquad b^2 = \left(\frac{l}{\epsilon}\right)^2 < 27m^2$$

Crosssection for capture:

$$\sigma = \pi b^2 = 27\pi m^2$$

Trajectories: goal: $r = r(\varphi)$
use $u = \frac{1}{r}$

$$\to \dot u = -\frac{1}{r^2}\dot r = -u^2 \dot r, \qquad \dot\varphi = l u^2$$
$$\to u' = \frac{du}{d\varphi} = \frac{\dot u}{\dot\varphi} = -\frac{\dot r}{l}$$

$$u'^2 = \frac{\epsilon^2}{l^2} - (1 - 2mu)\left(\frac{\mathcal{L}}{l^2} + u^2\right)$$

$$\frac{d}{d\varphi}: \qquad \cancel{2}\cancel{u'}u'' = \cancel{2}u\cancel{u'} + \frac{\cancel{2}m\mathcal{L}}{l^2}\cancel{u'} + \underbrace{\cancel{2}m3u^2\cancel{u'}}_{\text{GR}}$$

$$u'' + u - \mathcal{L}\frac{m}{l^2} = 3mu^2$$

i) Perihelion Advance

timelike geodesics: $\mathcal{L} = 1$ $\qquad$ τ = proper time

$$u'' + u - \frac{m}{l^2} = 3mu^2$$

(Non relativistic case: $u'' + u = \frac{m}{l^2}$

Solution:

$$u_0 = \tfrac{1}{d}(1 + \epsilon\cos\varphi) \qquad d = \frac{l^2}{m} = a(1-\epsilon^2)$$

$$(\epsilon > 0: \text{ perihelion at } \varphi = 0)$$

i.e.

$$r(\varphi) = \frac{1}{u_0} = \frac{d}{1+\epsilon\cos\varphi} \qquad \text{ellipse)}$$

pertubative ansatz: $u = u_0 + v$

1^{st} order in v (or m, while $\frac{m}{l^2}$ fixed)

$$v'' + v = \frac{3m}{d}\left(1 + 2\epsilon\cos\varphi + \epsilon^2\cos^2\varphi\right) \qquad ***$$

with initial condition $v(0) = v'(0) = 0$

$*\,*\,*$ is superposition of $v'' + v = \begin{cases} A_1 \\ A_2\cos\varphi \\ A_3\cos^2\varphi \end{cases}$

$$\longrightarrow v = \begin{cases} A_1 \\ \frac{1}{2}A_2\varphi\sin\varphi \\ A_3(\frac{1}{2} + \frac{1}{6}\cos 2\varphi - \frac{2}{3}\cos\varphi \end{cases}$$

only the 2^{nd} order contributes.

$$\Delta\varphi = -\frac{u'(2\pi)}{u''(2\pi)}$$

$$u'(2\pi) = \underbrace{u_0'(2\pi)}_{=0} + v'(2\pi) = A_2\pi = \frac{6\pi m\epsilon}{d^2}$$

$$U''(2\pi) = u_0''(2\pi) + \mathcal{O}(m) = -\frac{\epsilon}{d}$$

$$\Rightarrow \Delta\varphi = -\frac{u'(2\pi)}{u''(2\pi)} = \frac{6\pi m}{a(1-\epsilon^2)}$$

in full agreement with observations: Mercury $\Delta\varphi = 43''$ per century (after subtracting influence of other planets $\Delta\varphi = 591''$/century)

(apparent precession of equinoxes $\Delta\varphi = 5000''$/century)

ii) Deflection of light

null geodesics, $\mathcal{L} = 0$: $\qquad u'' + u = 3mu^2$

(unpertubed: $u'' + u = 0$, solutions $u_0 = b^{-1}\sin\varphi \;\rightarrow\; r = \frac{1}{u_0} = \frac{b}{\sin\varphi} \;\rightarrow\; r\sin\varphi = b$
(phase such that perihelian is at $\varphi = \frac{\pi}{2}$)
i.e. we have a straight line, as expected.)

pertubed: $u = u_0 + v$

$$v'' + v = 3mu_0^2 = 3mb^{-2}\sin^2\varphi$$
$$\text{with } v = 0, v' = 0 \text{ at } \varphi = \frac{\pi}{2}$$

solution:

$$\begin{aligned} u = u_0 + v &= \frac{\sin\varphi}{b} + \frac{3m}{b^2}\left(\frac{1}{2} + \frac{1}{6}\cos 2\varphi - \frac{1}{3}\sin\varphi\right) \\ &= \frac{\varphi}{b} + \frac{3m}{b^2}\left(\frac{2}{3} - \underbrace{\frac{1}{3}\varphi}_{\mathcal{O}(m)}\right) + \underbrace{\mathcal{O}(\varphi^2)}_{\mathcal{O}(m^2)} \end{aligned}$$

Zero shifted from $\varphi = 0$ to $\varphi = \varphi_\infty$

$$\varphi_\infty = -\frac{2m}{b}$$

Total deflection

$$\delta = 2|\varphi_\infty| = \frac{4m}{b} = \frac{1.75''}{b/R_o dot}$$

Experiment (1919, total eclipse)

$$\sphericalangle(A, B) \text{ increased by } 2\delta \text{ at eclipse}$$

7.4 The Kruskal extension: Black Hole

The singularity at $r = 2m$ is fake: failure of chart.
There is an extension of the Schwarzschild metric (Kruskal, 1960)

Kruskal transformation: $(u, v) \leftrightarrow (t, r), \quad \theta, \varphi$ fixed

$$u = \left(\frac{r}{2m} - 1\right)^{\frac{1}{2}} e^{\frac{r}{4m}} \cosh\left(\frac{t}{4m}\right)$$
$$v = \left(\frac{r}{2m} - 1\right)^{\frac{1}{2}} e^{\frac{r}{4m}} \sinh\left(\frac{t}{4m}\right)$$

We have ($\cosh^2 - \sinh^2 = 1$):

$$u^2 - v^2 = \left(\frac{r}{2m} - 1\right) e^{\frac{r}{2m}} = g\left(\tfrac{r}{2m}\right) \quad ****$$

$$\frac{v}{u} = \tanh\left(\frac{t}{4m}\right)$$

$$\{r < 2m\} \leftrightarrow \{|v| < u\}$$

Metric in new coordinates:

$$\left.\begin{aligned} g &= \left(1 - \frac{2m}{r}\right) dt^2 - \left(1 - \frac{2m}{r}\right)^{-1} dr^2 \\ &= \frac{32m^2}{r}\, e^{-\frac{r}{2m}} \underbrace{(dv^2 - du^2)}_{\substack{\text{conformally equiv. to Minkowski in } (u,v)\text{-space} \\ \to \text{ light cones at } 45^\circ}} \end{aligned}\right\} + \text{ angular part } (d\theta, d\varphi) \quad *****$$

with $r = r(u, v)$ as a solution of $****$

<u>Proof:</u> rescale $r = 4mr'$, $t = 4mt'$ (drop $'$, effectively $4m = 1$)

$$\begin{aligned} du &= \frac{\partial u}{\partial r} dr + \frac{\partial u}{\partial t} dt = 2r\,(2r-1)^{-\frac{1}{2}}\, e^r \cosh(t)\, dr + (2r-1)^{\frac{1}{2}}\, e^r \sinh(t)\, dt \\ dv &= 2r\,(2r-1)^{-\frac{1}{2}}\, e^r \sinh(t)\, dr + (2r-1)^{\frac{1}{2}}\, e^r \cosh(t)\, dt \\ dv^2 - du^2 &= (2r-1)e^{2r}(dt)^2 - 4r(2r-1)^{-1}e^{2r}(dr^2) \\ &= 2r\, e^{2r} \left[\left(1 - \tfrac{1}{2r}\right)(dt)^2 - \left(1 - \tfrac{1}{2r}\right)^{-1} (d^r)^2\right] \end{aligned}$$

The extension: $g(x)$ is monotonic increasing ($g' > 0$) for
$x \in (0, +\infty)\; r \longrightarrow g(x) \in (-1, +\infty)$

Hence $r = r(u, v)$ uniquelly determined by $****$ as long as

$$u^2 - v^2 > -1, \text{ i.e. } v^2 - u^2 < +1$$

So the metric defined by $*****$ extends from I to $I - IV$, still solving $R_{\mu\nu} = 0$!

<u>Remarks:</u>

1. On region II, introduce Schwarzschild coordinates ($t \in \mathbb{R}$, $r < 2m$)

$$u = \left(1 - \frac{r}{2m}\right)^{\frac{1}{2}} e^{\frac{r}{4m}} \sinh\left(\frac{t}{4m}\right)$$

$$v = \left(1 - \frac{r}{2m}\right)^{\frac{1}{2}} e^{\frac{r}{4m}} \cosh\left(\frac{t}{4m}\right)$$

$$\Longrightarrow v^2 - u^2 = \left(1 - \frac{r}{2m}\right) e^{\frac{r}{2m}}, \qquad \frac{u}{v} = \tanh\left(\frac{t}{4m}\right)$$

$$II = \{(u,v) | 0 < v^2 - u^2 < 1,\; v > 0\} \leftrightarrow \{(r,t) | 0 < r < 2m,\; t \in \mathbb{R}\}$$

$$\text{g} \quad \leftrightarrow \quad \text{Schwarzschild metric}$$
But: r is time coordinate
t is space coordinate

2. $v = u$ is an <u>event horizon</u>: boundary of the region[1] causally connected to distant observer

   Any particle entering II will reach singularity $r = 0$ within <u>finite</u> proper time

Timelike curve crossing horizon reaches singularity $r = 0$ in finite proper time:
$x(\lambda)$ (arbitrary parameter) with $\frac{du}{d\lambda}$, $\frac{dv}{d\lambda}$ finite
As $\lambda \to \underset{=0,\text{wlog}}{\underset{\downarrow}{\lambda_*}}$ (corresponding to $r = 0$)
Then

$$u^2 - v^2 = \mathcal{O}(\lambda)$$
$$r = \mathcal{O}(\lambda^{\frac{1}{2}})$$
$$\tau = \int^{\lambda_*=0} \frac{ds}{d\lambda} d\lambda \sim \int^{\lambda_*=0} r^{-\frac{1}{2}} d\lambda \sim \int^{\lambda_*=0} \lambda^{-\frac{1}{4}} < +\infty$$

Visualization: Equatorial plane $\theta = 0$ in the time slice $\{t = 0\}$ (2-dim) embedded in 3-dim Euclidean space (cyclic coordinates z, r, φ) as a graph $r = r(z)$ (surface of evolution)

fix φ

$$-ds^2 = \underbrace{\left(1 - \frac{2m}{r}\right)^{-1}}_{\frac{r}{r-2m}} \underbrace{dr^2}_{r'(z)^2 dz^2} = r'(z)^2 dz^2 + dz^2$$
$$r'(z)^2 \underbrace{\left(\frac{r}{r-2m} - 1\right)}_{\frac{2m}{r-2m}} = 1 \quad \Rightarrow \quad r'(z)^2 = \frac{r-2m}{2m}$$

solution:

$$r(z) = \frac{z^2}{8m} + 2m \qquad \left(\to r'(z) = \frac{z}{4m}\right)$$

Einstein-Rosen-Bridge

<u>Application:</u> Collapsing stars Star masses $0.07\ M_\odot < M \lesssim M_\odot$ End of thermonuclear evolution

- star may lose mass

- remaining mass $\begin{cases} M \lesssim 2M_\odot & \nearrow \; M \leq 1.4 M_\odot \quad \text{white dwarf} \\ & \searrow \; 1.4 M_\odot \leq M \leq 2M_\odot \quad \text{neutron star} \\ M \geq 2M_\odot & \text{black hole (nothing can sustain gravity)} \end{cases}$

<u>Theorem:</u> (Israel) Any static black hole is Schwarzschild (and hence spherical symmetric)

<u>Theorem:</u> (Birkhof) The most general solution of $R_{\mu\nu} = 0$ which is sphericylly symmetric (but not necessary static) is a piece of the Schwarzschild-Kruska metric

<u>Remark:</u> c.f. Newtonian gravity: spherically symmetric mass distribution (but not static):

$$\varphi(r) = -\frac{G_0 M}{r} \quad \rightarrow \text{ independent of } t$$

(M: total mass, const)

<u>Proof</u> (sketch): in suitable coordinates

$$ds^2 = e^{2a} dt^2 - \left(e^{2b} dr^2 + r^2 \left(d\theta^2 + \sin^2\theta d\varphi^2\right)\right)$$

$$\text{with } a = \underset{\substack{\uparrow \\ \text{new}}}{a(t,r)}, \quad b = \underset{\substack{\uparrow \\ \text{new}}}{b(t,r)}$$

Transformation compatible with Ansatz:

$$d\tilde{t} = e^{-c(t)} dt \quad \text{i.e. } t \mapsto \tilde{t} = \int^t e^{-c(s)} ds$$

$$e^{\tilde{a}(\tilde{t},r)} = e^{\tilde{a}(\tilde{t},r) - c(t)} dt \equiv e^{a(t,r)} dt$$

$$\implies \tilde{a}(\tilde{t}, r) = a(t,r) + c(t)$$

Ricci Tensor: non-zero components are:

$$\begin{aligned} R_{tt} &= R_{tt}^{(0)} - f \\ R_{rr} &= R_{rr}^{(0)} + e^{2(b-a)} f \\ R_{\theta\theta} &= R_{\theta\theta}^{(0)} + e^{2(b-a)} f \\ R_{\varphi\varphi} &= \left(\sin^2\theta\right) R_{\theta\theta} \\ R_{tr} &= R_{rt} = \frac{2\dot{b}}{r} \end{aligned}$$

(0) : static component

$f(t,r) = \dot{h}^2 - \dot{a}\dot{b} - \ddot{b}$

Field equation: $R_{\mu\nu} = 0$

$$R_{tr} = 0 \quad \rightarrow b = b(r)$$

$$R_{tt} e^{2(b-a)} + R_{rr} = 0 \underset{\substack{\text{as before} \\ \text{f drops out}}}{\longrightarrow} a' + b' = 0$$

$\rightarrow a(t,r) + b(r) = c(t)$ $\qquad$ $c(t) = 0$ wlog

$\rightarrow a(t,r) = a(r), \quad f = 0 \quad \rightarrow$ back to static case Schwarzschild metric

Back to collapse:

7.5 The Kerr metric and rotating black holes

Described by a stationary (rather than static) metric
Coordinates: (Boger-Lindquist)

$$t \in \mathbb{R}, \quad r > 0, \quad \theta, \quad \varphi \quad \text{spherical coordinates}$$

parameters: m, a

Notations:

$$\Delta = r^2 - 2mr - a^2$$
$$\rho^2 = r^2 + a^2 \cos^2\theta$$
$$\Sigma^2 = \left(r^2 + a^2\right)^2 - a^2 \Delta \sin^2\theta$$
$$(\text{identify: } \rho^4\Delta - 4mr^2a^2\sin^2\theta = \Sigma^2\left(\rho^2 - 2mr\right))$$

Metric: (Kerr 1963)

$$ds^2 = \left(1 - \frac{2mr}{\rho^2}\right) dt^2 + \frac{4mar}{\rho^2}\sin^2\theta \; dt \; d\varphi - \frac{\Sigma^2}{\rho^2}\sin^2\theta \; d\varphi^2 - \frac{\rho^2}{\Delta} dr^2 - \rho^2 d\theta^2$$

Alternate expression: complete $(d\varphi^2 + \dots)^2$

$$ds^2 = \frac{\rho^2}{\Sigma^2}\Delta \; dt^2 - \frac{\Sigma^2}{\rho^2}\sin^2\theta \; \left(d\varphi - \Omega \; dt^2\right) - \frac{\rho^2}{\Delta} dr^2 - \rho^2 \; d\theta^2$$

with

$$\Omega = a \, \frac{2mr}{\Sigma^2}$$

Remarks:

1. The special case $a = 0$: $\rightarrow$ Schwarzschild metric
   $\rightarrow \rho^2 = r^2, \Sigma^2 = r^4$
2. Kerr metric solves $R_{\mu\nu} = 0$
   It is the most general metric which is stationary and axisymmetric
3. any axisymmetric solution is given by Kerr (extension thereof) (c.f. Birkhof)

   any stationary black hole is given by Kerr (extension thereof) (c.f. Israel)

   "No Hair"theorem for black holes: There are caracterized by a, m (and nothing else) (& charge for $R_{\mu\nu} = T_{\mu\nu}^{textrmem}$)
4. Kerr $\underset{r\to\infty}{\longrightarrow}$ Minkowski (in polar oordinates)
5. Meaning of parameters:

   m: mass (from weak field limit at $r \to \infty$)

   $J = am$: angular momentum (without proof)

The metric has a singularity (g_{rr}) at $\Delta = 0$, i.e.

$$r = r_\pm = m \pm \sqrt{m^2 - a^2}$$

(exists only (and with it the black hole) for $|a| \leq m$)

$\rightarrow |J| \leq m^2$

Henceforth: $r > r_+$

The metric has Killing fields

$$\phi = \frac{\partial}{\partial \varphi}, \qquad K = \frac{\partial}{\partial t}$$

- ϕ is spacelike

$$(\phi, \phi) = g_{\varphi\varphi} < 0$$

- K is timelike

$$(K, K) = g_{tt} = \tfrac{1}{\rho^2}(r^2 + a^2 \cos^2\theta - 2mr) > 0$$

for

$$r > r_0(\theta) = m + \sqrt{m^2 - a^2 \cos^2\theta}$$

Meaning of ergosphere: different observers

in there: 4-velocity $u^\mu = (\dot{t}, \dot{r}, \dot{\theta}, \dot{\varphi})$ is timelike $(u, u) = +1 > 0$

i) <u>static observer</u> has fixed coordinates r, θ, φ:

$$u^\mu = (\dot{t}, 0, 0, 0) \propto K^\mu = (1, 0, 0, 0)$$

It can exist only for $r > r_0(\theta)$. For $r < r_0(\theta)$ any observer is dragged w.r.t. coordinate system

ii) <u>stationary observer</u> has fixed r, θ and

$$\omega \equiv \frac{d\varphi}{dt} = \frac{\dot{\varphi}}{\dot{t}} \qquad u^\mu = (\dot{t}, 0, 0, \omega \dot{t}) \propto (1, 0, 0, \omega)$$

$$(u, u) \propto \frac{\rho^2}{\Sigma^2}\,\Delta - \frac{\Sigma^2}{\rho^2} \sin\theta\,(\omega - \Omega)^2$$

u^μ is timelike

$$|\omega - \Omega| < \frac{\rho^2}{\Sigma^2} \frac{\Delta^{\frac{1}{2}}}{\sin\theta} \qquad (< \Omega \text{ if } r < r_0(\theta))$$

iii) freely falling observer starting from rest near infinity

Note: V Killing field, $x(\tau)$ geodesic. Then $(V;\dot{x})$ is constant in τ

By Noether: trajectory of $\mathcal{L} = \frac{1}{2}(\dot{x},\dot{x})$

conserved is: $V^\alpha \frac{\partial \mathcal{L}}{\partial \dot{x}^\alpha} = V^\alpha \dot{x}_\alpha$

Take $V = \phi$ and $u = \dot{x}$. At ∞ & rest: $(\phi, u) = 0$

At finite positions along the geodesics

$$0 = (\phi, u) = -\frac{\Sigma^2}{\rho^2}\sin^2\theta\,(\dot{\varphi} - \Omega\dot{t})(-\Omega)$$

$$\Rightarrow \frac{\dot{\varphi}}{\dot{t}} = \Omega = \frac{d\varphi}{dt} \qquad \text{angular velocity of drag}$$

$$= \frac{2mr}{\Sigma^2}a$$

Angular velocity at $r = r_+$:

$$\Sigma\big|_{r_+} = {r_+}^2 + a^2 = 2mr_+$$

$$\Omega_H = \Omega\Big|_{r=r_+} = a\,\frac{2mr}{\Sigma^2}\Big|_{r_+} = \frac{a}{2mr_+}$$

angular velocity of black hole

Energy extraction (Penrose 1969): Freely falling particle, $p = mu$ (momentum timelike)

„energy“ $E = (K, p)$ conserved

K timelike: $E > 0$

For observers near ∞: metric Minkowski, $K = (1,0,0,0)$ $E = p^+$

E is energy for that observer

particle decays

$$p = p_1 + p_2$$

$$E = E_1 + E_2 \qquad \text{with } E_i = (K, p_i)$$

2 gets out from ergosphere: $E_2 > 0$

possible: $E_1 < 0$

$$\Rightarrow E = E_1 + E_2 < E_2$$

Extracted energy:

$$E_2 - E > 0$$

7.6 Hawking radiation

Emission of energy is posiible even from a static black hole, provided quantum effects are taken into account: pair of particles created from nothing:

$$0 = p_1 + p_2$$

- outside of horizon: $0 = (K,p_1) + (K,p_2) = \underbrace{E_1}_{>0} + \underbrace{E_2}_{>0}$

- inside of horizon: either signs: possible but they do not get outside of horizon

Vakuum fluctuations produce 1 inside and 2 outside.
Discussion requires: QFT on curved spacetime.

a) Classical Klein-Gordon field

Action for scalar field φ of mass μ is

$$S = \int \underbrace{d^4x\sqrt{|g|}}_{\substack{\eta,\text{ Volume}\\ \text{of spacetime}}} \underbrace{\tfrac{1}{2}\left(\partial_\mu\varphi\partial^\mu\varphi - \mu^2\varphi^2\right)}_{\mathcal{L}} = \int dtL$$

$$\Rightarrow L = \int_{x^0=0} d^3x\sqrt{|g|}\mathcal{L}$$

$\partial^\mu\varphi = g^{\mu\nu}\partial_\nu\varphi$ arbitrary transformation $x \mapsto \tilde{x}$: $\tilde{\varphi}(\tilde{x}) = \varphi(x)$

$\longrightarrow$ S invariant

equations of motion

$$\partial_\nu \frac{\partial(\sqrt{|g|}\mathcal{L})}{\partial(\partial_\nu\varphi)} - \frac{\partial(\sqrt{|g|}\mathcal{L})}{\partial\varphi} = 0$$

is

$$\partial_\nu\left(\sqrt{|g|}g^{\mu\nu}\partial_\mu\varphi\right) + \mu^2\sqrt{|g|}\varphi = 0$$

$$\rightarrow \left(\Box_g + \mu^2\right)\varphi = 0 \qquad \Box_g = |g|^{-\frac{1}{2}}\partial_\nu(|g|g^{\mu\nu}\partial_\mu)$$

Conjugate momentum

$$\pi(x) = \sqrt{|g|}g^{\mu 0}(\partial_\mu\varphi)(x)$$

Hamiltonian

$$H = \int_{x^0=0} d^3x(\pi\,\partial_0\varphi - \mathcal{L})$$

$x = (x^0, \underline{x})$; initial data $\varphi(\underline{x}) = \varphi(x)\big|_{x^0=0}$, $\pi(\underline{x}) = \pi(x)\big|_{x^0=0}$

make up phase space

$$\Gamma = \left\{(\varphi(\underline{x}), \pi(\underline{x}))_{\underline{x}\in\mathbb{R}^3}\right\}$$

Poisson brackets

$$\{\pi(\underline{x}), \varphi(\underline{y})\} = \delta^{(3)}(\underline{x} - \underline{y})$$

Canonical equations of motion

$$\frac{\partial \varphi}{\partial t}(t, \underline{x}) = \{H, \varphi(t, \underline{x})\}$$
$$\frac{\partial \pi}{\partial t}(t, \underline{x}) = \{H, \pi(t, \underline{x})\}$$

$f,\ h$ complex solutions of Klein-Gordon equation

$$j^\mu = i g^{\mu\nu} \left(\overline{f}\ \partial_\nu h - (\partial_\nu \overline{f}) h\right)$$

Then

$$j^\mu{}_{;\mu}\sqrt{|g|} = \left(j^\mu \sqrt{|g|}\right)_{,\mu} \underset{\substack{\uparrow \\ \partial_\nu\left(\sqrt{|g|}g^{\mu\nu}\partial_\mu\varphi\right)+\mu^2\sqrt{|g|}\varphi=0}}{=} 0$$

Inner product on $K = \{\text{solutions } \ f(x) \text{of Klein-Gordon}\}$

$$\langle f, h\rangle = \int_\Sigma \sqrt{|g|} j^\mu \underset{\substack{\uparrow \\ \text{coordinate normal} \\ \text{of } \Sigma}}{d\sigma_\mu} = \int_{x^0=t} d^3x \sqrt{|g|} j^0$$

<u>independent</u> of Σ, resp. of t:

$$\left(\int_{\Sigma'} - \int_\Sigma\right) \sqrt{|g|} j^\mu d\sigma_\mu \underset{\substack{\uparrow \\ \text{Gauss for} \\ f,h\to 0 \text{ (spacelike)}}}{=} \int_V d^4x \left(\sqrt{|g|} j^\mu\right)_{,\mu} = 0$$

<u>Properties:</u>

$$\overline{\langle f, h\rangle} = -\langle \overline{f}, \overline{h}\rangle$$
$$\langle f, h\rangle = -\langle \overline{h}, \overline{f}\rangle$$

hence $(h = \overline{f})$

$$\langle f, \overline{f}\rangle = -\langle f, \overline{f}\rangle$$
$$= 0$$
$$\langle f, f\rangle = -\langle \overline{f}, \overline{f}\rangle$$

$\langle\cdot,\cdot\rangle$ not positive definit
but non-degenerate

$$\langle f,h\rangle = 0 \qquad \forall\, f\in K \qquad \Rightarrow \quad f=0$$

$$\langle f,h\rangle = i\int_{x^0=0} d^3x \left(\overline{f}\left(\sqrt{|g|}g^{0\nu}\partial_\nu h\right) - \left(\sqrt{|g|}g^{0\nu}\partial_\nu \overline{f}\right)h\right)$$

since $h\big|_{x^0=0}$, $\sqrt{|g|}g^{0\nu}\partial_\nu h\big|_{x^0=0}$ are arbitrary

Define functions $a(t)$ on Γ:

$$\begin{aligned} a(t) &= \langle f,\varphi\rangle \\ &= i\int_{x^0=0} d^3x \left(\overline{f}(\underline{x})\pi(\underline{x}) - \left(\sqrt{|g|}g^{0\nu}\partial_\nu\overline{f}\right)(\underline{x})\varphi(\underline{x})\right)\end{aligned}$$

Data $\underset{(f\in K)}{a(t)}$ determine $\varphi(\underline{x})$, $\pi(\underline{x})$ $a(t)$'s are not independent:

$$\overline{a(t)} = \overline{\langle f,\varphi\rangle} \underset{\varphi=\overline{\varphi}\in\mathbb{R}}{=} -a(\overline{t}) \qquad \clubsuit$$

$\overline{a}$, a's on equal footing

Poisson brackets

$$\{a(f),\overline{a}(h)\} = i\langle f,h\rangle$$

By $\clubsuit$

$$\{a(f),\underbrace{a(h)}_{-\overline{a(\overline{h})}}\} = -i\langle f,\overline{h}\rangle$$

$$\{\overline{a(f)},\overline{a(h)}\} = -i\langle \overline{f},h\rangle$$

b) Quantization of K.G.

$$\underset{\text{classical observer}}{a(f)} \longmapsto \underset{\text{quantum observer}}{a(f)}$$

with $a^*(f) = -a(\overline{f})$

$$\begin{aligned} i[a(f),a^*(h)] &= i\langle f,h\rangle \\ [a(f),a(h)] &= -\langle f,\overline{h}\rangle \\ [a^*(f),a^*(h)] &= -\langle \overline{f},h\rangle \end{aligned}$$

Algebra $\mathcal{A}$ generated by all $a(f)$ $(f\in K)$.

Quasi-free states ω on $\mathcal{A}$ specified by

(i)

$$\omega\left(a^*(f)a(h)\right) = \langle h, \rho f\rangle \qquad ♣♣$$

with ρ positive semi-definite on K, i.e.

$$\langle f, \rho f\rangle \geq 0$$

(ii) $\omega\left(\text{“} \prod a\text{'s } a^*\text{'s“}\right) =$ by Wick's Lemma: sum over all products of ♣♣

Then

$$a^*(\overline{f})a(\overline{h}) - a^*(h)a(f) = [a(f), a^*(h)] = \langle f, h\rangle$$

implies

$$\underbrace{\langle \overline{h}, \rho\overline{f}\rangle}_{=-\langle\overline{\rho}f,h\rangle} - \langle f, \rho h\rangle = \langle f, h\rangle \qquad \begin{aligned} \overline{\rho} &= C\rho C \\ C\rho\overline{f} &= C\rho C f = \overline{\rho} f \end{aligned}$$

<u>Particles & Antiparticles</u> $\mathcal{H} \subset K$ subspace such that

$$K = \mathcal{H} \oplus \overline{\mathcal{H}}$$

with $\overline{\mathcal{H}} = C\mathcal{H}$ and

$$\begin{aligned} \langle f, f\rangle &\geq 0 & & (f \in \mathcal{H}) \\ \langle f, h\rangle &= 0 & & (f \in \mathcal{H},\ h \in \overline{\mathcal{H}}) \end{aligned}$$

abstract: $\mathcal{H}$ 1-particle states
$\overline{\mathcal{H}}$ 1-antiparticle states

Examples of quasifree states for K.G.

$$\begin{aligned} \rho &= N \oplus N' & & \text{block diagonal w.r.t. } K = \mathcal{H} \oplus \overline{\mathcal{H}} \\ \text{with} \quad \langle f, Nf\rangle &\geq 0 & & \forall\, f \in \mathcal{H} \\ \overline{\rho} &= \overline{N}' \oplus \overline{N} & & \to N' = -1 - \overline{N} \end{aligned}$$

Example: $N = 0$

$$\omega\left(a^*(f)\, a(h)\right) = 0 \qquad \forall\, f, h \in \mathcal{H}$$

GNS Hilbert space is bosonic Fock space $\underset{\in\,\Omega}{\mathcal{F}}$ over $\mathcal{H}$ such that

$$a(f)\Omega = 0$$

$\mathcal{F}$ is spanned by

$$a^*(f_1)\dots a^*(f_n)\Omega \qquad (f_i \in \mathcal{H})$$

Indeed

$$\omega\left(a^*(f)a(h)\right) = (0) = \left(\Omega, a^*(f)a(h)\,\Omega\right) \qquad (f, h \in \mathcal{H})$$
$$\big(\text{all other expectation values e.g. for } h \in \overline{\mathcal{H}} \text{ follow}\big)$$

c) Quantization of K.G. in Minkowski space

Solutions $f \in K$ of $(\Box + \mu^2)f = 0$ are superpositions of plane waves

$$f = e^{i\left(\vec{k}\cdot\vec{x} \mp \omega t\right)}$$

with $\omega = \sqrt{\vec{k}^{\,2} + \mu^2} = \omega(\vec{k})$

$\mathcal{H} = \{\text{positive frequency solutions}\}$ satisfies requirements:

$K \ni f = f_+ \oplus f_-$ with $f_+ \in \mathcal{H}$, $f_- \in \overline{\mathcal{H}}$

$$\langle f, h\rangle = \int \frac{d^3k}{2\omega(\vec{k})}\Big(\overline{f_+(\vec{k})}h_+(\vec{k}) - \overline{f_-(\vec{k})}h_-(\vec{k})\Big)$$

with

$$f(x) = (2\pi)^{-\frac{3}{2}} \int \frac{d^3k}{2\omega(\vec{k})}\, f_\pm e^{-i\left(\vec{k}\cdot\vec{x} \mp \omega t\right)}$$

Note: $\mathcal{H}$ is Lorentz invariant

↑ positive frequency for observer at fixed $\vec{x}$

inertial observer, worldline $x^\mu(\tau) = u^\mu \tau + b^\mu$ $\qquad u^\mu, b^\mu$ fixed

$$e^{i\left(\vec{k}\cdot\vec{x} - \omega t\right)} = e^{ik_\mu x^\mu} = e^{-ik_\mu b^\mu} e^{-ik_\mu u^\mu \tau} \qquad (u,u) = +1$$

$$k^\mu(\omega, \vec{k}) \qquad \text{with} \qquad \begin{aligned} k_\mu u^\mu &> 0 \\ &= \omega u^0 - \vec{k}\cdot\vec{u} \\ &\geq \omega u^0 - \underbrace{|\vec{k}|}_{<\omega}\,\underbrace{|\vec{u}|}_{<u^0} > 0 \end{aligned}$$

Quantization of K.G. by picking “vakuum“

- Minkowski vakuum $\quad N = 0$

↑ Lorentz invariant

- positive temperature state (e.g. CMB)

$$\omega\left(a^*(f)\ a(h)\right) = \int \frac{d^3k}{2\omega(\vec{k})} \frac{1}{e^{3\omega(\vec{k})}-1} \overline{f(\vec{k})} h(\vec{k}) \qquad (f, h \in \mathcal{H})$$

$\omega\left(a^*(f)\ a(f)\right)$ expected number of particles in the 1-particle state $f \in \mathcal{H}$

f: wave packet concentrating at $\vec{k}_0$

$$\omega\left(a^*(f)\ a(f)\right) \longrightarrow \underbrace{\frac{1}{e^{3\omega(\vec{k}_0)}-1}}_{\text{thermal spectrum}} \underbrace{\langle f, f\rangle}_{=1}$$

This state is not Lorentz-invariant, since $\omega(\vec{k})$ is not.

Remark: In a curved spacetime with stationary metric ($\exists$ timelike Killing field) solutions have fixed frequency (or superpoitions thereof)

$$\mathcal{H} = \{\text{positive frequency}\}\ ?$$
$$N = 0\ ? \qquad \rightarrow \text{ Boulware vakuum}$$

Mathematically possible. But not physically correct.

d) Regge-Wheeler coordinates

New coordinates $(t, r_*, \theta, \varphi)$: transition from Schwarzschild coordinates, t, θ, φ fixed

$$r_* = r + 2m \log\left(\frac{r}{2m} - 1\right), \qquad \frac{dr_*}{dr} = 1 + \frac{1}{\frac{r}{2m}-1} = \left(1 - \frac{2m}{r}\right)^{-1}$$

Maps $r \in (2m, \infty) \mapsto r_* \in (-\infty, \infty)$ (tortoise coordinates)

r_* seems steadily going to $-\infty$ while in real world getting stuck at $r = 2m$

Metric

$$ds^2 = \left(1 - \frac{2m}{r}\right)\left(dt^2 - dr_*{}^2\right) + r^2(d\theta^2 + \sin^2\theta\ d\varphi^2) \qquad \text{with } r = r(r_*)$$

Consider particle radially infalling, crossing horizon ($t \to -\infty$, $r_* \to -\infty$) at proper time $\tau = 0$ (w.l.o.g.). There $r \simeq 2m$

$$\dot{r}^2 = -\epsilon^2 < 0, \qquad \frac{r-2m}{2m}\dot{t} = \epsilon$$

$$\rightarrow r - 2m = -\epsilon\tau, \qquad \dot{t} = -\frac{2m}{\tau} = \frac{dt}{d\tau}$$
$$t = -2m\log(-\tau) + const$$
$$r_* = 2m \log\left(-\frac{\epsilon\tau}{2m}\right) + 2m \quad (\tau \nearrow 0)$$

K.G. in Regge-Wheeler coordinates:

representations of angular-part-solution f of K.G.:

$$f(t, r_*, \theta, \varphi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{f_{lm}(t, r_*)}{r} Y_{lm}(\theta\varphi)$$

$\rightarrow$ K.G.

$$\left(\partial_t{}^2 - \partial_{r_*}{}^2 + V_l\right) f_{lm} = 0$$

$$V_l(r) = \left(1 - \frac{2m}{r}\right)\left(\frac{2m}{r^3} + \frac{l(l+1)}{r^2} + \mu^2\right)$$

has limits

$$V_{lm}(r) \rightarrow \quad \begin{cases} 0 & r_* \rightarrow -\infty \; (r \rightarrow 2m) \\ \mu^2 & r_* \rightarrow +\infty \; (r \rightarrow +\infty) \end{cases}$$

As $r_* \rightarrow -\infty$, solutions look like

$$f_{lm}(t, r_*) = f_{in}(t - r_*) + f_{out}(t + r_*)$$

f_{in} : incoming from white hole

f_{out} : outgoing to black hole

e) The expected number of outgoing particles (to $r \rightarrow +\infty$)

Consider wave packet f which

- consists of positive frequencies (peaked around ω)
- outgoing for $t \rightarrow +\infty$

For $r_* \rightarrow +\infty$ metric is Minkowski: f represents a particle at late times

$$n = \omega\left(a^*(f)\; a(f)\right) \qquad \text{occupation number of } f$$

What is ω? Equivalence principle suggests:

On states incoming from either $r_* = -\infty$ or $r_* = +\infty$ and to an observer in free fall there, ω in Minkowski-vacuum to him. (Unruh vacuum)

f is not of that form, but R and T are

$$\omega\left(a^*(R)\; a(R)\right) = 0$$

Since observer with $r = r_0$ ($r_0 \rightarrow +\infty$) is freely falling and R is of positive frequency

$$\rightarrow \quad \left.\begin{aligned} \omega\left(a^*(R)\; a(T)\right) = 0 \\ \omega\left(a^*(T)\; a(R)\right) = 0 \end{aligned}\right\} \text{ since } |\omega(A^*B)|^2 \leq \omega(A^*A)\omega(B^*B)$$

$$\Rightarrow \quad n = \omega\left(a^*(T)\, a(T)\right)$$

at $r_* \to -\infty$

$$T \propto e^{-i\omega(t-r_*)}$$

Freely falling observer approaching horizon:

$$t - r_* = -4m\log(-\tau) + const.$$
$$\rightarrow \quad T \propto \begin{cases} e^{4im\log(-\tau)} & \tau < 0 \\ 0 & \tau > 0 \end{cases}$$

$T = T_+ + T_-$ decomposition into $\pm$ frequencies

Unruh vacuum: $\omega\left(a^*(T_+)\, a(T_+)\right) = 0$

$$n = \omega\left(a^*(T_-)\, a(T_-)\right) = \langle T_-, \rho T_-\rangle = -\langle T_-, (1 + \underbrace{\overline{N_-}}_{=0})T_-\rangle$$
$$= -\langle T_-, T_-\rangle$$

T_+: positive frequency part

$$T_+(\tau) = \int_0^\infty d\omega \hat{T}_+(\omega) e^{-i\omega\tau}$$

is analytic in upper half plane in τ (T_- in the lower) ($\log z = log|z| + i\arg(z)$)

$$T_0(\tau) = e^{4im\log(-\tau)} = e^{4im\arg(-\tau)} \qquad (\tau < 0)$$

Analytic, const. to $\tau > 0$ through lower half plane

$$T_+ \stackrel{?}{=} c_+ \begin{cases} T_0(\tau) & \tau < 0 \\ T_0(\tau)e^{-4\pi m\omega} & \tau > 0 \end{cases}$$

$$T_- \stackrel{?}{=} c_- \begin{cases} T_0(\tau) & \tau < 0 \\ T_0(\tau)e^{+4\pi m\omega} & \tau > 0 \end{cases}$$

$$c_+ + c_- = 1$$
$$T = T_+ + T_- \Leftrightarrow c_+ e^{-4\pi m\omega} + c_- e^{+4\pi m\omega} = 0 \qquad (\tau > 0)$$
$$\Rightarrow c_\pm = \frac{1}{1 - e^{\mp 8\pi m\omega}}, \qquad \tilde{T}(\tau) = T(-\tau)$$
$$T_-(\tau) = c_-\left(T(\tau) + e^{4\pi m\omega}\tilde{T}(\tau)\right)$$
$$\Rightarrow \quad \langle T_-, T_-\rangle = |c_-|^2\left(1 - e^{8\pi m\omega}\right)\langle T, T\rangle \quad \left(\langle \tilde{T}, \tilde{T}\rangle = -\langle T, T\rangle = -\langle T_-, T_-\rangle\right)$$
$$= \frac{\langle T, T\rangle}{1 - e^{8\pi m\omega}} \qquad \left(\langle \underbrace{T}_{\substack{=0\\ \text{for}\\ \tau<0}}, \underbrace{\tilde{T}}_{\substack{=0\\ \text{for}\\ \tau>0}}\rangle = 0 \quad \text{no overlap}\right)$$

$$\Longrightarrow \quad n = \frac{\langle T, T\rangle}{e^{8\pi m\omega} - 1}$$

occupation number of outgoing state peaked at frequency ω (**Hawking Radiation**)

Apart from $\langle T, T\rangle$ (which depends on f and ω) this is black body radiation of temperature

$$\beta^{-1} = \frac{1}{8\pi m} = \frac{\hbar c^3}{8\pi G_0 M}$$

($\hbar, c = 1$in QFT)
($G_0 M = m$)

8 Linearized Gravity

8.1 The linearized field equations

Metric which, in suitable coordinates, is

$$\begin{aligned}
g_{\mu\nu} &= \eta_{\mu\nu} + h_{\mu\nu} && \eta: \text{ Minkowski}\\
h_{\mu\nu} &= h_{\nu\mu} && |h_{\mu\nu}| << 1\\
\Gamma^{\alpha}{}_{\mu\nu} &= \tfrac{1}{2} g^{\alpha\beta}\left(g_{\mu\beta,\nu} + g_{\nu\beta,\mu} - g_{\mu\nu,\beta}\right)\\
&= \tfrac{1}{2}\eta^{\alpha\beta}\left(h_{\mu\beta,\nu} + h_{\nu\beta,\mu} - h_{\mu\nu,\beta}\right) + \mathcal{O}(\not{h}^2)\\
&= \tfrac{1}{2}\left(h^{\alpha}{}_{\mu,\nu} + h^{\alpha}{}_{\nu,\mu} - h_{\mu\nu}{}^{,\alpha}\right)\\
R^{\alpha}{}_{\mu\beta\nu} &= \Gamma^{\alpha}{}_{\nu\mu,\beta} - \Gamma^{\alpha}{}_{\beta\mu,\nu} + \underbrace{\mathcal{O}(\Gamma^2)}_{=\mathcal{O}(h^2)=0}\\
R_{\mu\nu} &= R^{\alpha}{}_{\mu\alpha\nu} = \tfrac{1}{2}\left(-\Box h_{\mu\nu} - h_{,\mu\nu} + h^{\alpha}{}_{\mu,\alpha\nu} + h^{\alpha}{}_{\nu,\alpha\mu}\right) && \text{with } h = h^{\alpha}{}_{\alpha}
\end{aligned}$$

Convenient: trace reversed pertubation

$$\begin{aligned}
\gamma_{\mu\nu} &= h_{\mu\nu} - \tfrac{1}{2}\eta_{\mu\nu} h\\
\gamma = \gamma^{\alpha}{}_{\alpha} &= \underbrace{h^{\alpha}{}_{\alpha}}_{=h} - \tfrac{1}{2} 4h = -h\\
h_{\mu\nu} &= \gamma_{\mu\nu} - \tfrac{1}{2}\eta_{\mu\nu}\gamma
\end{aligned}$$

so

$$\begin{aligned}
G_{\mu\nu} &= R_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} R\\
[&= \tfrac{1}{2}\left(-\Box\gamma_{\mu\nu} - \eta_{\mu\nu}\gamma^{\alpha\beta}{}_{,\alpha\beta} + \gamma^{\alpha}{}_{\mu,\alpha\nu} + \gamma^{\alpha}{}_{\nu,\alpha\mu}\right) = kT_{\mu\nu}
\end{aligned}$$

Field equations ($G_{\mu\nu} = kT_{\mu\nu}$), linearized.
Remarks:

1. $2\, G\mu\nu_{,\nu} = -\Box\gamma^{\mu\nu}{}_{,\nu} - \eta^{\mu\nu}\gamma^{\alpha\beta}{}_{,\alpha\beta\nu} + \underbrace{\gamma^{\alpha\mu}{}_{,\alpha}{}^{\nu}{}_{\nu}}_{\Box\,\gamma^{\alpha\mu}{}_{\alpha}} + \gamma^{\alpha\nu}{}_{,\alpha}{}^{\mu}{}_{\nu} = 0$

   linearized 2[nd] Bianchi: $G^{\mu\nu}{}_{;\nu} = 0$

   $\rightarrow$ integr. condition for LFE: $T^{\mu\nu}{}_{,\nu} = 0$

2. Lorentz transformation $x^\mu \mapsto \Lambda^\mu{}_\nu x^\nu + a^\mu$ with $\Lambda_\mu{}^\sigma \Lambda_\nu{}^\tau \eta_{\sigma\tau} = \eta_{\mu\nu}$

$$h_{\mu\nu} \mapsto \Lambda_\mu{}^\sigma \Lambda_\nu{}^\tau h_{\sigma\tau}$$
$$\gamma_{\mu\nu} \mapsto \Lambda_\mu{}^\sigma \Lambda_\nu{}^\tau \gamma_{\sigma\tau}$$

   That makes the LFE form invariant.

3. Remark 2 does not mean "gravity + SR are compatible"; at least not if Equavalence Principle has to hold true.

   To be ruled out:

   metric (distances!) is giben by (a)$\eta_{\mu\nu}$ of (b)$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$

   (a) dust: equation of continuity $(\rho u^\mu)_{,\mu} = 0$

$$0 = T^{\mu\nu}{}_{,\nu} = (\rho u^\mu u^\nu)_{,\nu} = u^\mu \underbrace{(\rho u^\nu)_{,\nu}}_{=0} + \rho u^\nu u^\mu{}_{,\nu}$$

   $\rightarrow \quad u^\nu u^\mu{}_{,\nu} = 0$ geodesic equation for $\eta_{\mu\nu}$:
   dust particles go straight w.r.t. $\eta_{\mu\nu}$, no attraction, no gravity

   (b)

$$\left.\begin{array}{ll} \text{EP:} & T^{\mu\nu}{}_{;\nu} = 0 \\ \text{LFE:} & T^{\mu\nu}{}_{,\nu} \end{array}\right\} \text{incompatible except for } \Gamma^\mu{}_{\nu\sigma} = 0$$

8.2 Gauge transformations and gauges

LFE are gauge covariant: is a result of general covariance of FE (covariant w.r.t. coordinate transformation, resp. diffeomorphisms)

$$\varphi:\ x \mapsto \overline{x}, \qquad g \mapsto \varphi^* g$$

LFE are covariant under "small" diffeomorphisms

$$\overline{x}^\mu = x^\mu + \xi^\mu(x) \qquad (\xi^\mu \text{arbitrary, but } \underbrace{\text{small}}_{\mathcal{O}(h)})$$

$$g \mapsto g + L_\xi(g) = \eta + h + L_\xi\eta + Lh + \frac{L_\xi h}{\mathcal{O}(h!)}$$

i.e.

$$\begin{aligned}
h_{\mu\nu} &\mapsto h_{\mu\nu} + \xi^\alpha \eta_{\mu\nu,\alpha} + \xi_{\mu\nu} + \xi_{\nu\mu} \\
\gamma_{\mu\nu} &\mapsto h_{\mu\nu} + \xi_{\mu,\nu} + \xi_{\nu,\mu} - \tfrac{1}{2}\eta_{\mu\nu}(h + 2\xi^\alpha{}_{,\alpha}) \\
\gamma_{\mu\nu} &\mapsto \gamma_{\mu\nu} + \xi_{\mu,\nu} + \cdots - \tfrac{1}{2}\eta_{\mu\nu}\xi^\alpha{}_{,\alpha} \\
T &\mapsto T + \underbrace{L_\xi(T)}_{\mathcal{O}(h^3)} \\
T_{\mu\nu} &\mapsto T_{\mu\nu} \\
\Gamma^\alpha{}_{\mu\nu} &\mapsto \Gamma^\alpha{}_{\mu\nu} + \xi^\alpha{}_{,\mu\nu} \\
R^\alpha{}_{\mu\beta\nu} &\mapsto R^\alpha{}_{\mu\beta\nu} + \xi^\alpha{}_{,\nu\mu\beta} - \xi^\mu{}_{,\beta\mu\nu} = R^\alpha{}_{\dots}
\end{aligned}$$

$\Rightarrow$ LFE is form covariant
(partial analogy to Electrodynamics

$$\text{(a): } A_\mu \mapsto A_\mu + \Xi_{,\mu} \qquad\qquad \text{(b): } F_{\mu\nu} \mapsto F_{\mu\nu})$$

Gauges: **restrict gauge freedom (a)**

successively

i) Hilbert Gauge

$$\gamma^{\mu\nu}{}_{,\nu} = 0 \qquad \text{(cf. Lorentz gauge } A^\mu{}_{,\mu} = 0) \qquad \spadesuit$$

Start with $\overline{\gamma}^{\mu\nu}$ arbitrary: $\gamma^{\mu\nu} = \overline{\gamma}^{\mu\nu} + \xi^{\mu\nu} + \xi^{\nu\mu} - \frac{1}{2}\eta^{\mu\nu}\xi^\alpha{}_\alpha$

solves $\spadesuit$ if

$$(\overline{\gamma}^{\mu\nu}{}_{,\nu} =) \qquad \overline{\gamma}^{\mu\nu}{}_{,\nu} + \underbrace{\xi^{\mu,\nu}{}_\nu}_{\Box\,\xi^\mu} + \underbrace{\xi^{\nu,\mu}{}_\nu - \underbrace{\eta^{\mu\nu}\xi^\alpha{}_{\alpha,\nu}}_{=\xi^\alpha{}_\alpha{}^{,\mu}}}_{=0}$$

i.e.

$$\Box\,\xi^\mu = -\overbrace{\gamma}^{\mu\nu}{}_{,\nu}$$

inhomogenious wave equation, can be solved for ξ^μ even for prescribed initial values $\xi^\mu(x^0 = 0, \vec{x})$, $\partial_0\xi^\mu(x^0 = 0, \vec{x})$

<u>Recall:</u> $\Box\, u = \underset{\uparrow \text{given}}{f}$ $\qquad U(x^0, \vec{x}), \quad D_0 u(\dots)$

$$\begin{aligned}
D(x) &= \frac{1}{4\pi r}\left(\delta(x^0 - r) - \delta(x^0 + r)\right), \qquad x = (x^0, \vec{x}),\ r = |\vec{x}| \\
u(t, \vec{x}) &= \int d^3y\, (D(t, \vec{x} - \vec{y})\partial_0 u(0, \vec{y}) + \partial_0 D(t - \vec{x} - \vec{y})u(0, \vec{y}))
\end{aligned}$$

Residual gauge transformation: $\Box\,\xi^\mu = 0$

LFE: $\qquad -\Box\,\gamma_{\mu\nu} = 2\kappa T_{\mu\nu}$

- consistent with $T^{\mu\nu}{}_{,\nu} = 0$
- Field $\gamma_{\mu\nu}$ propagates at speed of light

ii) In vacuum ($T_{\mu\nu} = 0$) of if $T^{\mu}{}_{\mu} = 0$ (e.g. electro-magnetic field)

$$\Box\, \gamma = 0$$

Traceless Gauge: $\quad \gamma = 0 \quad \spadesuit\spadesuit$

Starting from $\overline{\gamma}^{\mu\nu}$ (in Hilbert gauge), $\gamma^{\mu\nu}$ solves $\spadesuit\spadesuit$ in addition if

$$\gamma = \overline{\gamma} - 2\xi^{\alpha}{}_{,\alpha} = 0 \qquad \text{with } \Box\, \xi^{\mu} = 0, \quad \Rightarrow \xi^{\alpha}{}_{,\alpha} = \tfrac{1}{2}\overline{\gamma} \qquad \spadesuit\spadesuit\spadesuit$$

For such γ

$$0 = \Box\, \xi^{\alpha}{}_{,\alpha} = \tfrac{1}{2}\Box\, \overline{\gamma} = 0$$

$\spadesuit\spadesuit\spadesuit$ will hold, provided at $x^0 = 0$

$$\xi^{\alpha}{}_{,\alpha} = \tfrac{1}{2}\overline{\gamma}, \qquad \partial_0 \xi^{\alpha}{}_{,\alpha} = \tfrac{1}{2}\partial_0\overline{\gamma}$$

$$\xi^{0}{}_{,0} + \div \vec{\xi} = \tfrac{1}{2}\overline{\gamma} \qquad \underbrace{\xi^{0}{}_{,00}}_{\Delta\xi^0} + \div\, \dot{\vec{\xi}} = \tfrac{1}{2}\partial_0\overline{\gamma} \qquad \spadesuit\spadesuit\spadesuit\spadesuit$$

assign $\vec{\xi}$, $\dot{\vec{\xi}}$ arbitrarily

Poisson equation for $\xi^0(x^0 = 0)$, and $\xi^{0}{}_{,0}(x^0 = 0)$ given by $\spadesuit\spadesuit\spadesuit\spadesuit$

Still residual: γ^{μ}, $\quad \Box\xi^{\mu} = 0, \quad \xi^{\alpha}{}_{,\alpha} = 0$

Note: in the Traceless Gauge: $\gamma_{\mu\nu} = h_{\mu\nu}$

iii) transversal traceless gauge

$$h^{0\mu} = 0$$

can be achieved. In this gauge (resp. coordinates) the metric distribution is only in space $h_{ij} \neq 0$.

[TT gauge: $\Gamma^{\alpha}{}_{00} = \frac{1}{2}\left(h^{\alpha}{}_{0,0} + h^{\alpha}{}_{0,0} - h_{00}{}^{,\alpha}\right) = 0$]

8.3 Gravitational waves

In TT gauge: $h^{\mu 0} = 0, \quad h^{i}{}_{i} = 0, \quad h^{ij}{}_{,j} = 0$

LFE:
In vacuum

$$\Box\, h_{ij} = 0$$

Plane wave solutions

$$h_{ij} = h_{ij}(\vec{e}\cdot\vec{x} - t) \qquad \vec{e}: \text{direction of propagation}, \quad |\vec{e}| = 1,\ \vec{e}\cdot\vec{x} = e_j x^j$$

h_{ij} depends on $s \in \mathbb{R}$ only.
Does h_{ij} satisfy the gauge conditions (because if not it's useless, then $\Box\, h_{ij} = 0$ doesn't describe gravity)?

$$\frac{dh_{ij}}{ds} e_j = 0 \qquad \blacklozenge$$

<u>Motion of test particle:</u> Let $u^\mu = (1, \vec{0})$ initial 4-velocity (at rest in TT coordinates).

$$\frac{du^\mu}{d\tau} + \Gamma^\mu{}_{\nu\sigma} u^\nu u^\sigma = 0$$

solved by $u^\mu(\tau) = (1, \vec{0})$
world line $x^\mu(\tau) = (\tau, \underset{\substack{\uparrow \\ \text{fixed}}}{\vec{x}_0})$

$$\frac{dx^\mu}{d\tau} = u^\mu$$

nearby particles have <u>fixed coordinate differences</u> $m^\mu = (0, \underset{\substack{\uparrow \\ \text{fixed}}}{\vec{u}})$

Yet <u>distances change</u>

$$(n, n) = \underset{\substack{\uparrow \\ = \eta_{\mu\nu} + h_{\mu\nu}}}{g_{\mu\nu}} n^\mu n^\nu = -\vec{n}^{\,2} + h_{ij}(s) n^i n^j$$

$$\Box\, h_{ij} = \partial^\mu \partial_\mu h_{ij} = \partial^\mu \frac{dh_{ij}}{ds}(-e_\mu) = \frac{d^2 h_{ij}}{ds^2} \underbrace{e^\mu e_\mu}_{=0} = 0$$

Put differently: Coordinates:

$$\begin{aligned} \tilde{x}^\mu &= x^\mu + \tfrac{1}{2} h^\mu{}_\nu x^\nu \qquad\qquad (\rightarrow \tilde{x}^0 = x^0) \\ &= \left(\delta^\mu{}_\nu + \tfrac{1}{2} h^\mu{}_\nu(x)\right) x^\nu \end{aligned}$$

$$d\tilde{x}^\mu = \Bigg(\delta^\mu{}_\nu + \tfrac{1}{2} h^\mu{}_\nu(x) + \underbrace{\frac{1}{2} \frac{\partial h^\mu{}_{\underset{\uparrow i}{\alpha}}}{\partial x^\nu} x^{\underset{\uparrow i}{\alpha}}}_{\mathcal{O}(\vec{x}/\lambda)}\Bigg) dx^\nu$$

λ : typical lengthscale of $h_{\mu\nu}$,
e.g. wave length

Claim: In a neighbourhood of world line $x^\mu(\tau) = (\tau, \vec{0})$ the metric

$$\tilde{g}_{\mu\nu} = \eta_{\mu\nu} + \mathcal{O}(h^2) + \mathcal{O}(\vec{x}/\lambda)$$

In fact:

$$\begin{aligned} \eta_{\mu\nu} d\tilde{x}^\mu d\tilde{x}^\nu &= \eta_{\mu\nu} (\delta^\mu{}_\sigma + \tfrac{1}{2} h^\mu{}_\sigma)(\delta^\nu{}_\tau + \tfrac{1}{2} h^\nu{}_\tau) dx^\sigma dx^\tau \\ &= \underbrace{(\eta_{\sigma\tau} + h_{\sigma\tau})}_{g_{\sigma\tau}} dx^\sigma dx^\tau + \mathcal{O}(h^2) \end{aligned}$$

Hence coordinates $\tilde{x}^\mu$ <u>are</u> distances (up to $\mathcal{O}(h^2), \mathcal{O}(\vec{x}/\lambda)$)

$$\tilde{n}^i(t) = n^i + \tfrac{1}{2}h^i{}_j(s)n^j = n^i - \tfrac{1}{2}h_{ij}(s)n^j \qquad s = \vec{e}\cdot\vec{x} - t$$

$$\Delta\tilde{n}^i(t) = -\tfrac{1}{2}h_{ij}(s)\tilde{n}^j$$

For $\tilde{n}^j = e^j$: $\Delta\tilde{n}^i(t) = 0$ by TT-gauge

$\to$ Gravitational wave is <u>transversal</u>.

monochromatic waves: $h_{ij} = \epsilon_{ij}e^{i\omega s}$
(physical field is $\operatorname{Re} h_{ij}$) Amplitude ϵ_{ij} arbitrary complex with

$$\left.\begin{aligned} \epsilon_{ij} &= \epsilon_{ji}, \\ \epsilon^i{}_i &= 0, \\ \epsilon_{ij}e^j &= 0 \end{aligned}\right\} \text{define a 2-dim complex vector space}$$

Pick $\vec{e} = \vec{e}_3$ (3-direction). Then

$$\epsilon = \left(\begin{array}{cc|c} \epsilon_{11} & \epsilon_{12} & 0 \\ \epsilon_{21} & -\epsilon_{11} & 0 \\ \hline 0 & 0 & 0 \end{array}\right) = (\operatorname{Re}\epsilon) + i(\operatorname{Im}\epsilon)$$

$$\left.\begin{matrix} \operatorname{Re}\epsilon \\ \operatorname{Im}\epsilon \end{matrix}\right\} \text{symmetric, traceless, real}$$

$$\begin{aligned} \Delta\vec{n}(t) &= -\tfrac{1}{2}\operatorname{Re}\left(\epsilon e^{i\omega s}\right)\vec{n} \\ &= -\tfrac{1}{2}\left((\operatorname{Re}\epsilon)\cos\omega s - (\operatorname{Im}\epsilon)\sin\omega s\right) \\ &\underset{\vec{x}=0}{\overset{\uparrow}{=}} -\tfrac{1}{2}\left((\operatorname{Re}\epsilon)\cos\omega t - (\operatorname{Im}\epsilon)\sin\omega t\right) \end{aligned}$$

Special polarizations:

i) <u>linear polarization:</u> $\operatorname{Re}\epsilon \parallel \operatorname{Im}\epsilon$ (proportional to one another)
diagonal in the <u>same</u> real orthonormal eigenbasis $\vec{e}_1 \perp \vec{e}_2$

$$\epsilon = A\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad (a \in \mathbb{C})$$

$$\Delta\vec{n}(t) = \tfrac{1}{2}\begin{pmatrix} -n_1 \\ n_2 \end{pmatrix}\left[(\operatorname{Re}A)\cos\omega t + (\operatorname{Im}A)\sin\omega t\right]$$

ii) <u>circular polarization:</u> $\operatorname{Re}\epsilon \perp \operatorname{Im}\epsilon$ & of same "length"
w.r.t. $(\epsilon, \delta) = \sum_{i,j}\epsilon_{ij}\delta_{ij} = tr\ (\epsilon\delta)$

$$\operatorname{Re}\epsilon = \operatorname{Re}\begin{pmatrix} \epsilon_{11} & \epsilon_{12} \\ \epsilon_{12} & -\epsilon_{11} \end{pmatrix} \qquad \operatorname{Im}\epsilon = \pm\operatorname{Re}\begin{pmatrix} -\epsilon_{12} & \epsilon_{11} \\ \epsilon_{11} & \epsilon_{12} \end{pmatrix}$$

$$= \pm R_{\frac{\pi}{4}}(\operatorname{Re}\epsilon)R_{\frac{\pi}{4}}{}^T$$

where $R_\varphi = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}$, Rotation by φ

w.r.t. eigenbasis of $\mathrm{Re}\,\epsilon$

$$\mathrm{Re}\,\epsilon = A\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \mathrm{Im}\,\epsilon = \pm A\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \epsilon = A\begin{pmatrix} 1 & \pm i \\ \pm i & -1 \end{pmatrix}$$

$$(A \in \mathbb{R}) \qquad\qquad R_\varphi \epsilon {R_\varphi}^T = e^{\mp 2i\varphi}\epsilon$$

$\Rightarrow$ helicity of gravitational wave is ± 2 (e.m. wave ± 1)

Motion of test particle with $\vec{n}$ on unit circle:

If particles not in free fall: add any other forces to tidal forces

Gravitational wave detectors: LIGO,...

goal: sensitivity $\frac{\Delta n}{n} \approx 10^{-24}$

on earth waves exp. with $\frac{\Delta n}{n} \lesssim 10^{-21}$

Emission of gravitational waves: Source $T^{\mu\nu}$ localized in space

$$\Box\, \gamma^{\mu\nu} = -2\kappa T^{\mu\nu} \qquad\qquad D_{\text{ret}}(x) = \frac{1}{4\pi r}\delta(x^0 - r)$$

$$r = |\vec{x}|$$

retarded solution

$$\begin{aligned} \gamma^{\mu\nu}(x) &= -\frac{2\kappa}{4\pi}\int d^4y\; D_{\text{ret}}(x-y) T^{\mu\nu}(y) \\ &= -\frac{2\kappa}{2\pi}\int d^3y \frac{T^{\mu\nu}(\vec{y}, t - |\vec{x}-\vec{y}|)}{|\vec{x}-\vec{y}|} \end{aligned}$$

For $r \gg d \ll \lambda$ (d: diameter of source; λ: wave length)

$\gg \lambda$

$$\gamma^{\mu\nu} = -\frac{\kappa}{2\pi r}\underbrace{\int d^3y\; T^{\mu\nu}(\vec{y}, \underbrace{t-r}_{s})}_{\epsilon^{\mu\nu}(s)}$$

$$I = \frac{\kappa}{360\pi c^5}\, tr\, \dddot{Q}^2 \qquad \text{(Einstein 1917)}$$

I: Intensity: energy emitted per unit time in all directions

$$Q_{ij}(t) = \int d^3y\; T^{00}(\vec{y}, t)(3y_i y_j - \delta_{ij}\vec{y}^{\,2})$$