General Relativity
HS 08

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Inhaltsverzeichnis

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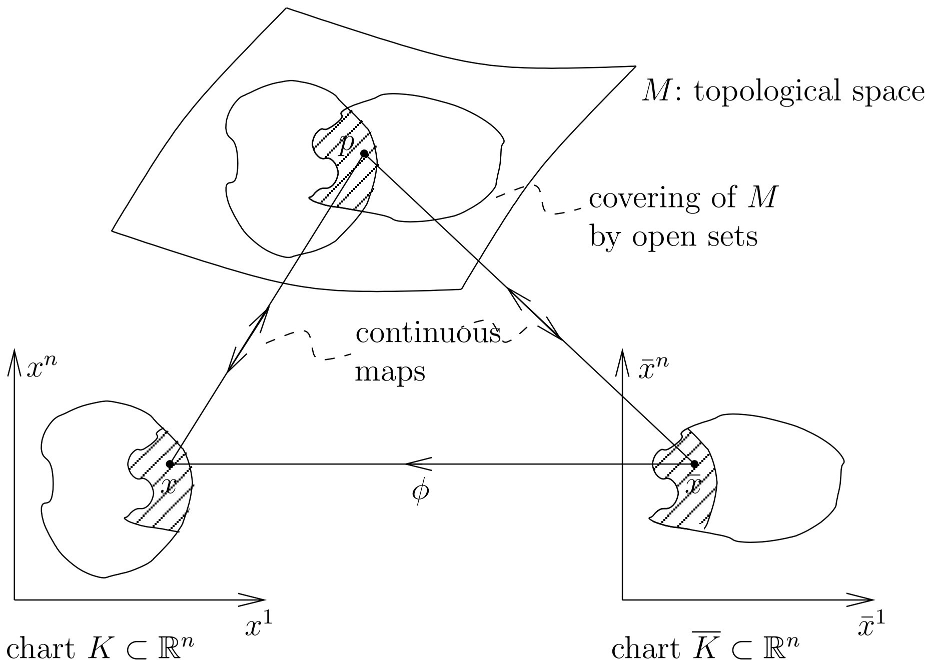
1. Manifolds and tensor fields

1. Differentiable manifolds

A **differentiable manifold** M is "locally homeomorphic to $\mathbb{R}^n$", meaning it is defined by the following elements:



Within the shaded overlap region of two charts the change of coordinates $x \leftrightarrow \bar{x}$ (transition functions) are differentiable any number of times. Definition: $\dim M = n$.

Notions

- Differentiable functions $f : M \to \mathbb{R}$ (algebra $\mathcal{F} = C^\infty(M)$)
- $\mathcal{F}_p$: algebra of C^∞-functions defined in any neighborhood of p ($f = g$ means $f(q) = g(q)$ in some neighborhood of p)
- Differentiable curve $\gamma : \mathbb{R} \to M$
- Differentiable map: $M \to M'$

The notions are to be understood my means of a chart: e.g. $f : M \to \mathbb{R}$ is differentiable if $x \mapsto f(p(x)) \equiv f(x)$ is. This is independent of the chart representing a neighborhood of p.

Tangent space T_p of the point $p \in M$

A **vector** $X \in T_p$ is a linear map $\mathcal{F}_p \to \mathbb{R}$ with the derivation property

$$X(fg) = (Xf)g(p) + f(p)(Xg) \ . \tag{1.1}$$

T_p is a linear space. In any chart (representing p) we have

$$Xf = X^i f_{,i}(x) \ : \qquad X^i = X(x^i) \ ,$$

where $_{,i} = \partial/\partial x^i$ and $x^i \in \mathcal{F}_p$ denotes the coordinate function $p \mapsto x^i$.

Proof: For $f \equiv 1$ we have $f^2 = f$, whence $Xf = 2Xf = 0$. Thus $Xf = 0$, if f is constant. Let p have coordinates $x = 0$. The identity

$$f(x) = f(0) + x^i \underbrace{\int_0^1 dt f_{,i}(tx)}_{g_i(x)}$$

implies by (1.1) $Xf = X(x^i) \cdot g_i(0) = X^i f_{,i}(0)$. □

Directional derivative

Let $\gamma(t) \in M$ be a curve through $\gamma(0) = p$. Then γ defines an $X \in T_p$ through

$$Xf = \frac{d}{dt} f(\gamma(t))\Big|_{t=0} \ , \tag{1.2}$$

denoted by $X = \dot{\gamma}(0)$. In components:

$$X^i = \frac{d\gamma^i}{dt}\Big|_{t=0}$$

(γ^i = coordinates of γ). One can thus regard a tangent vector X as an equivalence class of curves through p.

Basis of T_p

T_p has dimension n. In any basis $(e_1, \dots e_n)$ we have

$$X = X^i e_i \ .$$

Change of basis:

$$\bar{e}_i = \phi_i{}^k e_k \ ; \qquad \bar{X}^i = \phi^i{}_k X^k \tag{1.3}$$

inverse-transposed

In particular $e_i = \partial/\partial x^i$ is called **coordinate** basis (w.r.t. a chart). Upon change of chart,

$$\phi_i{}^k = \frac{\partial x^k}{\partial \bar{x}^i} \ ; \qquad \phi^i{}_k = \frac{\partial \bar{x}^i}{\partial x^k} \ . \tag{1.4}$$

The cotangent space T_p^*

Dual space of T_p: a **covector** $\omega \in T_p^*$ is a linear form

$$\omega : X \mapsto \omega(X) \equiv \langle \omega, X \rangle \in \mathbb{R} \ .$$

In particular, for any $f \in \mathcal{F}_p$

$$df : X \mapsto Xf$$

is an element of T_p^*. The elements $df = f_{,i} dx^i$ form a linear space of dimension n, hence all of T_p^*.

Basis $(e^1, \dots e^n)$ of T_p^*:

$$\omega = \omega_i e^i \ .$$

In particular the **dual basis** (of a basis $(e_1, \dots e_n)$ of T_p) is given by

$$\langle e^i, X \rangle = X^i \ , \qquad \text{or} \ \ \langle e^i, e_k \rangle = \delta^i{}_k \ .$$

Thus $\omega_i = \langle \omega, e_i \rangle$. Upon changing the basis the ω_i transforms like the e_i and the e^i like the X^i (cf. (1.3)). In particular we have for the coordinate basis

$$e_i = \frac{\partial}{\partial x^i} \ ; \qquad e^i = dx^i \ .$$

The change of basis then is

$$\frac{\partial}{\partial \bar{x}^i} = \frac{\partial x^k}{\partial \bar{x}^i} \frac{\partial}{\partial x^k} \ ; \qquad d\bar{x}^i = \frac{\partial \bar{x}^i}{\partial x^k} dx^k \ .$$

Tensors on T_p

Tensors are multilinear forms on T_p^* and T_p, e.g. a tensor T of type $\binom{1}{2}$ (for short: $T \in \otimes_2^1 T_p$): $T(\omega, X, Y)$ is a trilinear form on $T_p^* \times T_p \times T_p$. In particular $\otimes_1^0 T_p = T_p^*$, $\otimes_0^1 T_p = (T_p^*)^* \cong T_p$, as well as $\otimes_0^0 T_p = \mathbb{R}$. The **tensor product** is defined between tensors of any type, e.g.

$$T(\omega, X, Y) = R(\omega, X) \cdot S(Y) \ : \qquad T = R \otimes S \ .$$

Components (w.r.t. a pair of dual bases)

$$T(\omega, X, Y) = \underbrace{T(e^i, e_j, e_k)}_{\equiv T^i{}_{jk}} \ \underbrace{\omega_i X^j Y^k}_{e_i(\omega) e^j(X) e^k(Y)} \ ,$$

hence

$$T = T^i{}_{jk} e_i \otimes e^j \otimes e^k \ .$$

Any tensor of this type can therefore be obtained as a linear combination of tensor products $X \otimes \omega \otimes \omega'$ with $X \in T_p$, ω, $\omega' \in T_p^*$, denoted as $\otimes_2^1 T_p = T_p \otimes T_p^* \otimes T_p^*$.

Change of basis

$$\overline{T}^i{}_{jk} = T^\alpha{}_{\beta\gamma} \phi^i{}_\alpha \phi_j{}^\beta \phi_k{}^\gamma \ . \tag{1.5}$$

Trace

Any bilinear form $b \in T_p^* \otimes T_p$ determines a linear form $l \in (T_p \otimes T_p^*)^*$ such that

$$\begin{tikzcd} T_p \times T_p^* \arrow[r, "b"] \arrow[d, "\otimes"] & \mathbb{R} \\ T_p \otimes T_p^* \arrow[ur, "l"'] & \end{tikzcd}$$

$$l(X \otimes \omega) = b(X, \omega) \ .$$

Proof: The map $l \mapsto b$ is one-to-one and on grounds of dimension also onto. $\square$

In particular $\operatorname{tr} T$ is a linear form on tensors T of type $\binom{1}{1}$ defined by

$$\operatorname{tr}(X \otimes \omega) = \langle \omega, X \rangle \ .$$

In components w.r.t. a dual pair of bases we have

$$\operatorname{tr} T = T^i{}_i \ .$$

Similarly,

$$T^i{}_{jk} \mapsto S_k = T^i{}_{ik}$$

defines for instance a map from tensors of type $\binom{1}{2}$ to tensors of type $\binom{0}{1}$.

The tangent map

Let φ be a differentiable map $M \to \overline{M}$; let $p \in M$ and $\bar{p} = \varphi(p)$. Then φ induces a linear map

$$\varphi_* : T_p(M) \to T_{\bar{p}}(\overline{M}) \ ,$$

which we describe in two ways:

(a) For any $\bar{f} \in \mathcal{F}_{\bar{p}}(\overline{M})$ set

$$(\varphi_* X)\bar{f} = X(\bar{f} \circ \varphi) \ .$$

(b) Let γ be a representative of X (cf. (1.2)). Then let

$$\overline{\gamma} = \varphi \circ \gamma$$

be a representative of $\varphi_* X$. This agrees with (a), because

$$\frac{d}{dt} \bar{f}(\overline{\gamma}(t))\Big|_{t=0} = \frac{d}{dt} (\bar{f} \circ \varphi)(\gamma(t))\Big|_{t=0} \ .$$

W.r.t. bases $(e_1, \dots e_n)$ of T_p, $(\bar{e}_1, \dots, \bar{e}_{\bar{n}})$ of $T_{\bar{p}}$ reads $\overline{X} = \varphi_* X$

$$\overline{X}^i = (\varphi_*)^i{}_k X^k$$

with $(\varphi_*)^i{}_k = \langle \overline{e}^i, \varphi_* e_k \rangle$ or, in case of coordinate bases,

$$(\varphi_*)^i{}_k = \frac{\partial \overline{x}^i}{\partial x^k} \ .$$

The **adjoint map** φ^* of φ_* is

$$\varphi^* : T^*_{\bar{p}} \to T^*_p \ ; \qquad \overline{\omega} \mapsto \varphi^* \overline{\omega}$$

with

$$\langle \varphi^* \overline{\omega}, X \rangle = \langle \overline{\omega}, \varphi_* X \rangle \ .$$

The same result is obtained from the definition

$$\varphi^* : d\bar{f} \mapsto d(\bar{f} \circ \varphi) \ , \qquad (\bar{f} \in \mathcal{F}(\overline{M})) \ . \tag{1.6}$$

In components, $\omega = \varphi^* \overline{\omega}$ reads

$$\omega_k = \overline{\omega}_i (\varphi_*)^i{}_k \ .$$

From now on we limit ourselves to (local) diffeomorphisms, i.e. maps φ such that φ^{-1} exists in an neighborhood of $\bar{p}$.

$$\dim M = \dim \overline{M} \; ; \qquad \det\Big(\frac{\partial \overline{x}^i}{\partial x^k}\Big) \neq 0 \; .$$

Then φ_* and φ^* are invertible and may be extended to tensors of arbitrary type.

Example. Type $\binom{1}{1}$:

$$\begin{aligned} (\varphi_* T)(\overline{\omega}, \overline{X}) &= T(\varphi^* \overline{\omega}, \varphi_*^{-1} \overline{X}) \; , \\ (\varphi^* \overline{T})(\omega, X) &= \overline{T}(\varphi^{*-1} \omega, \varphi_* X) \; . \end{aligned}$$

Here, φ_*, φ^* are each other's inverse and we have

$$\begin{gathered} \varphi_*(T \otimes S) = (\varphi_* T) \otimes (\varphi_* S) \; , \\ \operatorname{tr}(\varphi_* T) = \varphi_*(\operatorname{tr} T) \end{gathered} \tag{1.7}$$

(tr = any trace) and similarly for φ^*. In components $\overline{T} = \varphi_* T$ reads

$$\overline{T}^i{}_k = T^\alpha{}_\beta \frac{\partial \overline{x}^i}{\partial x^\alpha} \frac{\partial x^\beta}{\partial \overline{x}^k} \tag{1.8}$$

(coordinate basis). This is formally the same as the transformation (1.5) when changing basis.

2. Tensor fields

A **vector field** on M is a linear map $X : \mathcal{F} \to \mathcal{F}$ with the derivation property

$$X(fg) = (Xf)g + f(Xg) \; . \tag{1.9}$$

This implies that $(Xf)(p)$ depends only on the equivalence class $f \in \mathcal{F}_p$. **Proof:** From $f = 0$ in a neighborhood U of p we conclude by means of a function g with $\operatorname{supp} g \subset U$, $g(p) = 1$, that $(Xf)(p) = 0$. $\square$

Hence, for any $p \in M$

$$X_p : f \mapsto (Xf)(p)$$

is a vector in T_p. In a chart we thus have

$$(Xf)(x) = X^i(x) f_{,i}(x) \; , \qquad \text{d.h.} \qquad X = X^i(x) \frac{\partial}{\partial x^i}$$

with smooth components $X^i(x)$: vector fields are linear differential operators of first order. The vector fields on M form a linear space on which the following operations are defined as well

$$\begin{aligned} X &\mapsto fX && \text{(multiplication by } f \in \mathcal{F}) \; , \\ X, Y &\mapsto [X, Y] = XY - YX && \text{(commutator)} \; . \end{aligned}$$

Indeed, $[X,Y]$, unlike XY, satisfies (1.9):

$$\begin{aligned}[X,Y](fg) &= X((Yf)g + f(Yg)) - Y((Xf)g + f(Xg)) \\ &= ([X,Y]f)g + f([X,Y]g) \ .\end{aligned}$$

Moreover the Jacobi identity holds true

$$[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0 \ . \tag{1.10}$$

1-forms are "f-linear" maps

$$\omega : X \mapsto \omega(X) \in \mathcal{F}$$

from the space of vector fields to $\mathcal{F}$, i.e.,

$$\omega(fX) = f\omega(X) \ ,$$

besides of linearity. This implies that $\omega(X)(p)$ depends only on X_p **Proof:** chart: $p \in U \to \mathbb{R}^n$, $p \mapsto x = 0$. Let $\operatorname{supp} f \subset U$, $f(p) = 1$. If $X_p = 0$, then $\omega(X)(p) = \omega(f^2X)(p) = (fX^i)(0)\omega(f\partial/\partial x^i) = 0$, since $X^i(0) = 0$. □

Thus, for any $p \in M$ a covector $\omega_p \in T_p^*$ is defined through

$$\omega(X)(p) = \langle \omega_p, X_p \rangle$$

In any chart we then have

$$\omega(X) = \omega_i(x)X^i(x) \ , \qquad \text{i.e.} \qquad \omega = \omega_i(x)dx^i$$

($dx^i : X \mapsto X^i$, locally) with smooth components $\omega_i(x)$.

Tensor fields

Example: A tensor field R of type $\binom{1}{2}$ is a function $R(\omega, X, Y)$ of: ω (1-form), X, Y (vector fields), taking values in $\mathcal{F}$, which is f-linear in each variable. A tensor field can also be viewed as a function

$$R : p \in M \mapsto R_p : \text{tensor on } T_p \ ,$$

which is smooth in terms of its components: In any chart we have

$$R(\omega, X, Y) = R^i{}_{jk}(x)\omega_i(x)X^j(x)Y^k(x)$$

with smooth components $R^i{}_{jk}(x)$. They transform according to (1.5, 1.4) under coordinate changes.

Tangent map

($\varphi : M \to \overline{M}$ differentiable)

1-forms: $\overline{\omega} \mapsto \varphi^*\overline{\omega}$. The 1-form $\varphi^*\overline{\omega}$ on M is defined by (1.6) and f-linearity. Equivalently,

$$(\varphi^*\overline{\omega})_p = \varphi^*\overline{\omega}_{\varphi(p)} \ .$$

Let henceforth φ be a diffeomorphism.

Vector fields: $X \mapsto \varphi_* X$, a vector field on $\overline{M}$:

$$(\varphi_* X)\overline{f} = [X(\overline{f} \circ \varphi)] \circ \varphi^{-1} ,$$

hence $(\varphi_* X)_{\bar{p}} = \varphi_* X_{\varphi^{-1}(\bar{p})}$. We have

$$\varphi_*[X, Y] = [\varphi_* X, \varphi_* Y]$$

Tensor fields: $\overline{R} \to \varphi^* \overline{R}$, $(\varphi_* = \varphi^{*-1})$, e.g. $\overline{R}$ of type $\binom{1}{1}$:

$$(\varphi^* \overline{R})(\omega, X) = \overline{R}(\varphi^{*-1}\omega, \varphi_* X) \circ \varphi ,$$

resp.

$$(\varphi^* \overline{R})_p = \varphi^* R_{\varphi(p)} ,$$

i.e. φ^* acts pointwise on the tensors of the field.

Flows and generating vector fields

A **flow** is a 1-parameter group of diffeomorphisms $\varphi_t : M \to M$, $t \in \mathbb{R}$ with

$$\varphi_t \circ \varphi_s = \varphi_{t+s} .$$

In particular $\varphi_0 = \mathrm{id}$. Moreover the **orbits** (or integral curves) of any point $p \in M$

$$t \mapsto \varphi_t(p) \equiv \gamma(t)$$

shall be differentiable. A flow determines a **vector field** X by means of

$$Xf = \frac{d}{dt}(f \circ \varphi_t)\Big|_{t=0} , \tag{1.11}$$

$$\text{d.h.} \qquad X_p = \frac{d}{dt}\gamma(t)\Big|_{t=0} = \dot{\gamma}(0) ,$$

where $\dot{\gamma}(0)$ is the tangent vector to γ at the point $p = \gamma(0)$. At the point $\gamma(t)$ we then have

$$\dot{\gamma}(t) = \frac{d}{dt}\varphi_t(p) = \frac{d}{ds}(\varphi_s \circ \varphi_t)(p)\Big|_{s=0} = X_{\varphi_t(p)} .$$

i.e. $\gamma(t)$ solves the ordinary differential equations

$$\dot{\gamma}(t) = X_{\gamma(t)} ; \qquad \gamma(0) = p . \tag{1.12}$$

The generating vector field thus determines the flow uniquely. (In general a vector field many not generate a flow, because (1.12) may not admit global solutions (i.e. for all $t \in \mathbb{R}$). For most purposes "local flows" suffice, though.)

3. The Lie derivative

The derivative of a vector field V rests on the comparison of V_p and $V_{p'}$ at nearly points p, p'. Since $V_p \in T_p$, $V_{p'} \in T_{p'}$ belong to different spaces their difference can be taken only after $V_{p'}$ has been transported to T_p. This can be achieved by means of the tangent map φ_* (Lie transport).

The **Lie derivative** $L_X R$ of a tensor field R in direction of a vector field X is defined by

$$L_X R = \frac{d}{dt} \varphi_t^* R \Big|_{t=0} \tag{1.13}$$

or, somewhat more explicitely,

$$(L_X R)_p = \frac{d}{dt} \varphi_t^* R_{\varphi_t(p)} \Big|_{t=0} .$$

Here, φ_t is the (local) flow generated by X, whence $\varphi_t^* R_{\varphi_t(p)}$ is a tensor on T_p depending out. In order to express L_X in components we write φ_t in a chart

$$\varphi_t : x \mapsto \bar{x}(t, x)$$

and linearize in small t:

$$\bar{x}^i = x^i + tX^i(x) + \dots , \qquad x^i = \bar{x}^i - tX^i(\bar{x}) + \dots ,$$

hence

$$\frac{\partial^2 \bar{x}^i}{\partial x^k \partial t} = -\frac{\partial^2 x^i}{\partial \bar{x}^k \partial t} = X^i{}_{,k}$$

at $t = 0$. As an example, let R be of type $\binom{1}{1}$. By (1.8) we then have

$$(\varphi_t^* R)^i{}_j(x) = R^\alpha{}_\beta(\bar{x}) \frac{\partial x^i}{\partial \bar{x}^\alpha} \frac{\partial \bar{x}^\beta}{\partial x^j} .$$

Taking a derivative w.r.t. t at $t = 0$ yields:

$$(L_X R)^i{}_j = R^i{}_{j,k} X^k - R^\alpha{}_j X^i{}_{,\alpha} + R^i{}_\beta X^\beta{}_{,j} . \tag{1.14}$$

Properties of L_X

(a) L_X is a linear map from tensor field to tensor fields of the same type
(b) $L_X(\operatorname{tr} T) = \operatorname{tr}(L_X T)$, (tr any trace)
(c) $L_X(T \otimes S) = (L_X T) \otimes S + T \otimes (L_X S)$
(d) $L_X f = Xf, (f \in \mathcal{F})$
(e) $L_X Y = [X, Y]$, (Y: vector field)

Proof: (a) follows from (1.13), (b,c) from (1.7), (d) from (1.11) and (e) from

$$\begin{aligned}(L_X Y)f &= \Big(\frac{d}{dt} \varphi_t^* Y \Big|_{t=0}\Big) f = \frac{d}{dt} (\varphi_{-t*} Y) f \Big|_{t=0} = \frac{d}{dt} Y\,(f \circ \varphi_{-t}) \circ \varphi_t \Big|_{t=0} \\ &= Y\Big(\frac{d}{dt} f \circ \varphi_{-t} \Big|_{t=0}\Big) + \frac{d}{dt} (Y(f) \circ \varphi_t) \Big|_{t=0} = -YXf + XYf .\end{aligned}$$

$\square$

Alternate definition of L_X: For a given vector field X the properties (a–e) (which do not refer to flows) determine $L_X R$ uniquely for any tensor field R. In particular, this definition agrees with (1.13).

Proof: Because of (c) we just need to show that $L_X\omega$ is defined for 1-forms ω. This follows from

$$(L_X\omega)(Y) = \mathrm{tr}(L_X\omega \otimes Y) = \mathrm{tr}\, L_X(\omega \otimes Y) - \mathrm{tr}\,\omega \otimes L_X Y = X\omega(Y) - \omega([X,Y]) \ .$$

$\square$

Further properties of L_X

L_X is linear in X (but not f-linear!) and

$$L_{[X,Y]} = L_X L_Y - L_Y L_X \ .$$

Proof: The r.h.s. satisfies (a–c) and agrees with the l.h.s. on $f \in \mathcal{F}$, as well as on vector fields Z, the latter because of (1.10).

On the meaning of $[X,Y]=0$

Let φ_t be the flow generated by X. If $[X,Y]=0$, then

$$\varphi_{t*}Y = Y \ , \qquad \text{i.e.} \qquad Y_{\varphi_t(p)} = \varphi_{t*}Y_p \ , \tag{1.15}$$

since

$$\frac{d}{dt}\varphi_t^* Y = \frac{d}{ds}\varphi_{t+s}^* Y\Big|_{s=0} = \varphi_t^*\Big(\frac{d}{ds}\varphi_s^* Y\Big|_{s=0}\Big) = \varphi_t^*([X,Y]) = 0 \ .$$

Let now ψ_s be the flow generated by Y. By (1.15) we have

$$\frac{d}{ds}\varphi_t(\psi_s(p)) = \varphi_{t*}Y_{\psi_s(p)} = Y_{\varphi_t(\psi_s(p))} \ ,$$

i.e. $\varphi_t(\psi_s(p))$ satisfies the ODE and the initial value for $\psi_s(\varphi_t(p))$. Hence they are the same. The result is:

$$[X,Y]=0 \quad \Longleftrightarrow \quad \varphi_t \circ \psi_s = \psi_s \circ \varphi_t \tag{1.16}$$

(if X, Y generate global flows).

4. Differential forms

A p-**form** Ω is a totally antisymmetric tensor field of type $\binom{0}{p}$:

$$\Omega(X_{\pi(1)}, \dots, X_{\pi(p)}) = (\mathrm{sgn}\,\pi)\Omega(X_1, \dots X_p)$$

for any permutation π of $\{1, \dots, p\} : \pi \in S_p$, with $\mathrm{sgn}\,\pi$ being its parity. In particular, $\Omega \equiv 0$ for $p > \dim M$. Any tensor field of type $\binom{0}{p}$ can be antisymmetrized by means of the operation $\mathcal{A}$:

$$(\mathcal{A}T)(X_1, \dots, X_p) = \frac{1}{p!}\sum_{\pi \in S_p}(\mathrm{sgn}\,\pi)T(X_{\pi(1)}, \dots, X_{\pi(p)}) \ . \tag{1.17}$$

We have $\mathcal{A}^2 = \mathcal{A}$. The **exterior product** of a p_1-form Ω^1 with a p_2-form Ω^2 is the (p_1+p_2)-form

$$\Omega^1 \wedge \Omega^2 = \frac{(p_1+p_2)!}{p_1!p_2!}\mathcal{A}(\Omega^1 \otimes \Omega^2). \tag{1.18}$$

Properties:

$$\begin{aligned}
&\Omega^1 \wedge \Omega^2 = (-1)^{p_1 p_2}\, \Omega^2 \wedge \Omega^1 \\
&\Omega^1 \wedge (\Omega^2 \wedge \Omega^3) = (\Omega^1 \wedge \Omega^2) \wedge \Omega^3 = \frac{(p_1+p_2+p_3)!}{p_1!p_2!p_3!}\mathcal{A}(\Omega^1 \otimes \Omega^2 \otimes \Omega^3)
\end{aligned}$$

Components: in a (local) basis of 1-forms $(e^1, \ldots e^n)$

$$\begin{aligned}
\Omega &= \Omega_{i_1\ldots i_p}\, e^{i_1} \otimes \ldots \otimes e^{i_p} = \mathcal{A}\Omega \\
&= \Omega_{i_1\ldots i_p}\, \mathcal{A}(e^{i_1} \otimes \ldots \otimes e^{i_p}) \\
&= \Omega_{i_1\ldots i_p}\, \frac{1}{p!}\, e^{i_1} \wedge \ldots \wedge e^{i_p} \\
&= \Omega_{i_1\ldots i_p}\, e^{i_1} \wedge \ldots \wedge e^{i_p} \qquad \text{(when restricting the sum to } i_1 < \ldots < i_p\text{)}.
\end{aligned} \tag{1.19}$$

Examples:

For 1-forms A, B we have

$$(A \wedge B)_{ik} = A_i B_k - A_k B_i\ .$$

For a 2-form A and 1-form B,

$$(A \wedge B)_{ikl} = A_{ik}B_l + A_{kl}B_i + A_{li}B_k\ , \tag{1.20}$$

because

$$A \wedge B = A_{ik}B_l \frac{1}{2} e^i \wedge e^k \wedge e^l = \underbrace{(A_{ik}B_l + \text{zykl.})}_{(A\wedge B)_{ikl}} \frac{1}{6} e^i \wedge e^k \wedge e^l\ ,$$

since the bracket is totally antisymmetric.

The exterior derivative of a differential form

The derivative df of a 0-form $f \in \mathcal{F}$ is the 1-form $df(X) = Xf$: the argument X acts as a derivation. The derivative $d\Omega$ of a 1-form Ω is

$$d\Omega(X_1, X_2) = X_1\Omega(X_2) - X_2\Omega(X_1) - \Omega([X_1, X_2])\ .$$

The last term ensures that $d\Omega$ is a 2-form, being f-linear in X_1, X_2:

$$\begin{aligned}
d\Omega(fX_1, X_2) &= fX_1\Omega(X_2) - X_2\Omega(fX_1) - \Omega([fX_1, X_2]) \\
&= fX_1\Omega(X_2) - \big((X_2 f)\Omega(X_1) + fX_2\Omega(X_1)\big) - \Omega(f[X_1, X_2] + (X_2 f)X_1) \\
&= f d\Omega(X_1, X_2)\ .
\end{aligned} \tag{1.21}$$

On $\Omega \wedge f = f\Omega$ the product rule $d(\Omega \wedge f) = d\Omega \wedge f - \Omega \wedge df$ applies, since

$$\begin{aligned}
d(\Omega \wedge f)(X_1, X_2) &= X_1(f\Omega)(X_2) - X_2(f\Omega)(X_1) - (f\Omega)(X_1, X_2) \\
&= f d\Omega(X_1, X_2) - \Omega(X_1)f(X_2) + \Omega(X_2)f(X_1)\ .
\end{aligned} \tag{1.22}$$

Moreover we have $d^2 f = 0$, because

$$\begin{aligned} d^2 f(X_1, X_2) &= X_1 df(X_2) - X_2 df(X_1) - df([X_1, X_2]) \\ &= X_1 X_2 f - X_2 X_1 f - [X_1, X_2] f = 0 \ . \end{aligned} \tag{1.23}$$

The generalization of the definition to p-forms Ω is

$$\begin{aligned} d\Omega(X_1 \dots X_{p+1}) &= \sum_{i=1}^{p+1} (-1)^{i-1} \Omega(X_1 \dots \widehat{X}_i, \dots X_{p+1}) \\ &\quad + \sum_{i<j} (-1)^{i+j} \Omega([X_i, X_j], X_1, \dots, \widehat{X}_i, \dots, \widehat{X}_j, \dots X_{p+1}) \ , \end{aligned} \tag{1.24}$$

where $\widehat{\ }$ denotes omission. Analogously to (1.21-1.23) one shows the

Properties of d

(a) d is a liner map from p-forms to $(p+1)$-forms
(b) $d(\Omega^1 \wedge \Omega^2) = d\Omega^1 \wedge \Omega^2 + (-1)^{p_1} \Omega^1 \wedge d\Omega^2$
(c) $d^2 = 0$, d.h $d(d\Omega) = 0$
(d) $df(X) = Xf$, $(f \in \mathcal{F})$

Alternate definition of d: By means of (a–d), hence without reference to commutators.

Proof: We need to show that d is defined on all p-forms Ω. By (1.19) we have w.r.t. a coordinate basis

$$\Omega = \frac{1}{p!} \Omega_{i_1 \dots i_p} \, dx^{i_1} \wedge \dots \wedge dx^{i_p} \ , \tag{1.25}$$

hence

$$d\Omega = \frac{1}{p!} d\Omega_{i_1 \dots i_p} \wedge dx^{i_1} \wedge \dots \wedge dx^{i_p} \ .$$

□

Components: $({}_{,i} = \partial/\partial x^i)$

$$\begin{aligned} p! d\Omega &= \Omega_{i_1 i_2 \dots i_p, i_0} dx^{i_0} \wedge \dots \wedge dx^{i_p} \\ &= -\Omega_{i_0 i_2 \dots i_p, i_1} dx^{i_0} \wedge \dots \wedge dx^{i_p} \\ &= (-1)^k \Omega_{i_0 \dots \hat{\imath}_k \dots i_p, i_k} dx^{i_0} \wedge \dots \wedge dx^{i_p} \ , \qquad (k = 0, \dots p) \ , \\ d\Omega &= \underbrace{\sum_{k=0}^{p} (-1)^k \Omega_{i_0 \dots \hat{\imath}_k \dots i_p, i_k}}_{(d\Omega)_{i_0 \dots i_p}} \frac{1}{(p+1)!} dx^{i_0} \wedge \dots \wedge dx^{i_p} \ . \end{aligned} \tag{1.26}$$

Examples:

$p = 1$: $$(d\Omega)_{ik} = \Omega_{k,i} - \Omega_{i,k} \ , \tag{1.27}$$

$p = 2$: $$(d\Omega)_{ikl} = \Omega_{ik,l} + \Omega_{kl,i} + \Omega_{li,k} \ . \tag{1.28}$$

Further properties: For any map $\varphi : M \to N$,

$$\varphi^* \circ d = d \circ \varphi^* \ . \tag{1.29}$$

Proof: Because of (1.25, 1.7) and property (b) it suffices to verify (1.29) on:

0-forms $\bar{f}$: (1.29) is identical to (1.6);
1-forms, which are differentials $d\bar{f}$: because of (c) we have

$$(\varphi^* \circ d)(d\bar{f}) = 0 \; , \qquad (d \circ \varphi^*)(d\bar{f}) = d(\varphi^* \circ d\bar{f}) = (d^2 \circ \varphi^*)(\bar{f}) = 0 \; .$$

$\square$

Setting $\varphi = \varphi_t$ (the flow generated by X) and forming $d/dt\big|_{t=0}$, one obtains the infinitesimal version of (1.29):

$$L_X \circ d = d \circ L_X \; . \tag{1.30}$$

Definition: A p-Form ω with

- $\omega = d\eta$ is **exact**;
- $d\omega = 0$ is **closed**.

The implication "ω exact $\Rightarrow \omega$ closed" holds true, but the converse generally not. A local converse is the Poincaré lemma:

Lemma: Let $G \subset M$ be an open domain in a "star-shaped" chart. Any point in the chart is connected to the origin by a straight line lying in the chart. Let ω be a p-form with $d\omega = 0$ in G. Then there exists a $(p-1)$-form η such that

$$\omega = d\eta \; .$$

Proof: see p. 15.
Remark: Obviously, η is not unique, since "gauge transformations" $\eta \to \eta + d\rho$, with ρ any $(p-2)$-form, leave $d\eta$ unchanged.

The integral of an n-form

Let an **orientation** be given on M: an atlas of "positively oriented" charts, i.e.

$$\det\left(\frac{\partial \bar{x}^i}{\partial x^j}\right) > 0 \tag{1.31}$$

for any change of coordinates. (Not every manifold is orientable; example: the Möbius strip). An n-form ω, $(n = \dim M)$,

$$\omega = \omega_{i_1 \dots i_n} \frac{1}{n!} dx^{i_1} \wedge \dots \wedge dx^{i_n} = \underbrace{\omega_{1\dots n}}_{\omega(x)} dx^1 \wedge \dots \wedge dx^n$$

is determined by the single component $\omega(x)$; under a change of coordinates it transforms as

$$\bar{\omega}(\bar{x}) = \bar{\omega}_{1\dots n} = \omega_{i_1 \dots i_n} \frac{\partial x^{i_1}}{\partial \bar{x}^1} \dots \frac{\partial x^{i_n}}{\partial \bar{x}^n} = \omega(x) \det\left(\frac{\partial x^i}{\partial \bar{x}^j}\right) \; . \tag{1.32}$$

The **integral** of an n-form is defined as follows. If $\operatorname{supp}\omega$ is contained in a (positive) chart, we set

$$\int_M \omega = \int dx^1 \dots dx^n \omega(x^1 \dots x^n) \; .$$

For $\operatorname{supp}\omega$ in the intersection of two charts, $\int \omega$ is independent of the one used by (1.31, 1.32) and

$$\int dx^1 \dots dx^n \omega(x) = \int d\bar{x}^1 \dots d\bar{x}^n \omega(x) \left| \det\Bigl(\frac{\partial x^i}{\partial \bar{x}^j}\Bigr) \right| .$$

For arbitrary ω of compact support we define

$$\int_M \omega = \sum_k \int h_k \omega . \tag{1.33}$$

Here $\{h_k\}$ is a partition of unity on M:

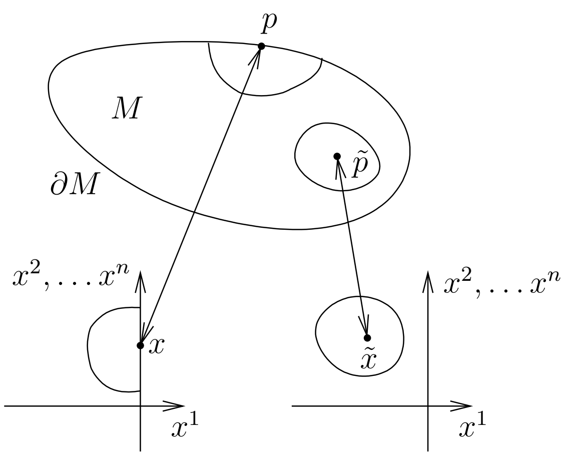
$$h_k \in \mathcal{F} ; \qquad h_k \geq 0 ; \qquad \sum_k h_k = 1$$

such that each $\operatorname{supp} h_k$ is contained in some chart (such partitions do exist). The independence of (1.33) on the choice of the partition is seen by considering the refinement $\{h_k g_l\}$ of two partitions $\{h_k\}$, $\{g_l\}$.

Remark: Upon reversing the orientation, $\int_M \omega$ changes sign.

The Stokes Theorem

A(n-dimensional) **manifold with boundary** is a locally homeomorphic to $\mathbb{R}^{n-} = \{(x^1 \dots x^n) \in \mathbb{R}^n \mid x^1 \leq 0\}$:



The boundary ∂M consists of those $p \in M$, whose image x in some (and hence any) chart satisfies $x^1 = 0$.

Orientation of the boundary: an orientation on M induces one on ∂M: If $(x^1 \dots x^n)$ is a positive chart for $U \subset M$, then $(x^2 \dots x^n)$ is one on $\partial M \cap U$. (Show the consistency of this definition.)

Stokes Theorem: Let M, $(\dim M = n)$, be an oriented manifold with boundary. Then, for any $(n-1)$-form ω:

$$\int_M d\omega = \int_{\partial M} \omega . \tag{1.34}$$

Proof: Let $\{h_k\}$ be a partition of unity on M. We decompose $\omega = \sum_k h_k \omega$. We then need to prove (1.34) in two special cases:

(a) $\operatorname{supp}\omega$ lies in a chart without boundary. Then (cf. (1.26))

$$\int_M d\omega = \int dx^1 \dots dx^n \sum_{k=1}^{n} (-1)^{k-1} \omega_{1\dots\hat{k}\dots n,k} = 0 .$$

(b) $\operatorname{supp}\omega$ lies in a chart with boundary. Then

$$\begin{aligned}\int_M d\omega &= \int dx^1 \dots dx^n \sum_{k=1}^{n} (-1)^{k-1} \omega_{1\dots\hat{k}\dots n,k} = \int dx^1 \dots dx^n \omega_{2\dots n,1} \\ &= \int dx^2 \dots dx^n \omega(0, x^2, \dots x^n) = \int_{\partial M} \omega \ ,\end{aligned}$$

since $(x^2 \dots x^n)$ is a positively oriented chart of ∂M. $\square$

The inner product of a p-form

Let X be a vector field on M. For any p-form Ω let

$$(i_X \Omega)(X_1, \dots, X_{p-1}) = \Omega(X, X_1, \dots, X_{p-1}) \ ,$$

($= 0$ if $p = 0$).

Properties

(a) i_X is a linear map from p-forms to $(p-1)$-forms
(b) $i_X(\Omega^1 \wedge \Omega^2) = (i_X\Omega^1) \wedge \Omega^2 + (-1)^{p_1} \Omega^1 \wedge i_X \Omega^2$
(c) $i_X^2 = 0$
(d) $i_X df = Xf$, $(f \in \mathcal{F})$
(e) $L_X = i_X \circ d + d \circ i_X$

Proof: (a–d) are straightforward. It suffices to verify (e) on:

0-forms f: both sides equal Xf.
1-form, which are differentials df: both sides equal $d(Xf)$ because of (1.30). $\square$

Applications:

1) **The Gauss Theorem:**
The manifold M is oriented iff there is an n-form η with $\eta_p \neq 0$ for all $p \in M$ ("volume form"). Let X be a vector field. Then $d(i_X\eta)$ is a n-form and a function $\mathrm{div}_\eta X \in \mathcal{F}$ is defined through

$$(\mathrm{div}_\eta X)\eta = d(i_X\eta)$$

(also $= L_X\eta$, because of (e)). The Stokes Theorem immediately implies the **Gauss Theorem**:

$$\int_M (\mathrm{div}_\eta X)\eta = \int_{\partial M} i_X \eta \ .$$

In a chart:

$$\begin{aligned}(i_X\eta)_{i_2\dots i_n} &= X^a \eta_{a\, i_2 \dots i_n} \\ d(i_X\eta)_{1\dots n} &= \sum_{k=1}^{n} (-1)^{k-1} \big(\underbrace{X^a \eta_{a\,1\dots\hat{k}\dots n}}_{(-1)^{k-1} X^k \eta_{1\dots n}}\big)_{,k} = \big(X^k \eta_{1\dots n}\big)_{,k}\end{aligned}$$

hence, setting again $\eta(x) \equiv \eta_{1\dots n}(x)$,

$$\mathrm{div}_\eta X = \frac{1}{\eta} (\eta X^k)_{,k} \ .$$

For the integral $\int_{\partial D} i_X\eta$ (only boundary charts contribute, see figure on p. 13) we obtain:

$$\int_{\partial M} i_X\eta = \int dx^2 \ldots dx^n (i_X\eta)_{2\ldots n}\,(0,x^2,\ldots,x^n) = \int dx^2 \ldots dx^n (\eta X^1)(0,x^2,\ldots,x^n)$$

because $(x^2,\ldots,x^n)$ is a positively oriented chart of ∂M.

2) **Proof** of the Poincaré lemma: By using a chart we may assume $U \subset \mathbb{R}^n$ and thus identify $T_x \cong \mathbb{R}^n$. We shall construct a map T from p- to $(p-1)$-forms on U with

$$(T \circ d + d \circ T)\omega = \omega$$

(ω: arbitrary p-form). For $d\omega = 0$ this implies $d\eta = \omega$ for $\eta = T\omega$, as claimed. Construction of T:

$$(T\omega)_x = \int_0^1 t^{p-1}(i_X\omega)_{tx}dt\ , \qquad (x \in U)\ ,$$

where X is the vector field with components $X^i(x) = x^i$. Then (e) implies

$$[(Td + dT)\omega]_x = \int_0^1 t^{p-1}(L_X\omega)_{tx}\,dt\ . \tag{1.35}$$

Here $L_X\omega = (x\nabla)\omega + p\omega$ because by (1.14) we have

$$(L_X\omega)_{i_1\ldots i_p} = x^k\omega_{i_1\ldots i_p,k} + \sum_{p=1}^{p} \underset{\substack{\uparrow \\ j\text{-th position}}}{\omega_{i_1\ldots k\ldots i_p}}\ \underbrace{X^k{}_{i_j}}_{\delta^k{}_{i_j}}\ .$$

Moreover we have $[(x\nabla)\omega]_{tx} = tx(\nabla\omega)_{tx} = t\frac{d}{dt}\omega_{tx}$, hence

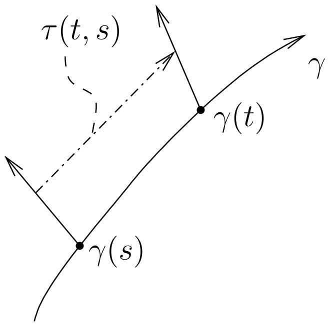
$$t^{p-1}(L_X\omega)_{tx} = t^p\frac{d}{dt}\omega_{tx} + pt^{p-1}\omega = \frac{d}{dt}(t^p\omega_{tx})$$

and (1.35) equals ω_x. $\square$

2. Affine connections

1. Parallel transport and covariant derivative

Definition: Any curve γ in M is equipped with a parallel transport of vectors.



$$\tau(t,s) : T_{\gamma(s)} \to T_{\gamma(t)}$$

is a linear map with

$$\tau(t,t) = 1, \qquad \tau(t,s)\tau(s,r) = \tau(t,r) \ . \tag{2.1}$$

In any chart we have

$$\frac{\partial}{\partial t}\tau^i{}_k(t,s)\Big|_{t=s} = -\Gamma^i{}_{lk}(\gamma(s))\dot\gamma^l(s) \ . \tag{2.2}$$

Remarks: 1) The Lie transport φ_{t*} along an orbit of Y is **not** of the form (2.2):

$$\frac{d}{dt}(\varphi_{t*})^i{}_k\Big|_{t=0} = Y^i{}_{,k} \ .$$

2) A parallel transported vector $X(t) = \tau(t,s)X(s) \in T_{\gamma(t)}$ solves, in a chart, the differential equation

$$\dot X^i(s) = -\Gamma^i{}_{lk}(\gamma(s))\dot\gamma^l(s)X^k(s) \ . \tag{2.3}$$

The $\dot X^i$ are not the components of a vector, hence the **Christoffel symbols** $\Gamma^i{}_{lk}(x)$ not those of a tensor (s. below).

3) Equation (2.3) states, that the $\dot X^i$ are linear in $\dot\gamma^l$, X^k. Because of this property (which is independent of the chart) $\tau(t,s)$ does not depend on the parameterization of γ (but also not just on the endpoints $\gamma(s)$, $\gamma(t)$).

4) Because of (2.1) we also have

$$\frac{\partial}{\partial s}\tau^i{}_k(t,s)\Big|_{s=t} = \Gamma^i{}_{lk}(\gamma(t))\dot\gamma^l(t) \ . \tag{2.4}$$

5) Upon changing chart,

$$\overline{\tau}^i{}_k(t,s) = \tau^p{}_q(t,s)\frac{\partial \bar x^i}{\partial x^p}\Big|_{\gamma(t)}\frac{\partial x^q}{\partial \bar x^k}\Big|_{\gamma(s)} \ .$$

Applying $\frac{\partial}{\partial s}\big|_{s=t}$ and (2.4) implies

$$\overline{\Gamma}^i{}_{lk}\dot{\bar\gamma}^l = \Gamma^p{}_{rq}\underbrace{\dot\gamma^r}_{\frac{\partial x^r}{\partial \bar x^l}\dot{\bar\gamma}^l}\frac{\partial \bar x^i}{\partial x^p}\frac{\partial x^q}{\partial \bar x^k} + \delta^p{}_q\frac{\partial \bar x^i}{\partial x^p}\frac{\partial^2 x^q}{\partial \bar x^k \partial \bar x^l}\dot{\bar\gamma}^l \ ,$$

hence:

$$\overline{\Gamma}^i{}_{lk}(\overline{x}) = \Gamma^p{}_{rq}(x)\frac{\partial \bar x^i}{\partial x^p}\frac{\partial x^q}{\partial \bar x^k}\frac{\partial x^r}{\partial \bar x^l} + \frac{\partial \bar x^i}{\partial x^p}\frac{\partial^2 x^p}{\partial \bar x^k \partial \bar x^l} \ . \tag{2.5}$$

Conversely, a field $\Gamma^i{}_{lk}(x)$ with this transformation law determines a parallel transport along any curve $\gamma(t)$ by means of the differential equation (2.2).

The **parallel transport** is extended to tensors by means of the requirements

$$
\begin{aligned}
&\tau(t,s)(T\otimes S) = (\tau(t,s)T)\otimes(\tau(t,s)S)\ ,\\
&\tau(t,s)(\operatorname{tr} T) = \operatorname{tr}(\tau(t,s)T)\ , \qquad (\operatorname{tr} = \text{any trace})\\
&\tau(t,s)c = c\ , \qquad (c\in\mathbb{R})\ ,
\end{aligned}
$$

so e.g. for a covector ω

$$\langle \tau(t,s)\omega, \tau(t,s)X\rangle_{\gamma(t)} = \langle \omega, X\rangle_{\gamma(s)}$$

and for a tensor T of type $\binom{1}{1}$

$$(\tau(t,s)T)(\tau(t,s)\omega,\tau(t,s)X) = T(\omega,X)\ . \tag{2.6}$$

In components:

$$(\tau(t,s)T)^i{}_k = T^\alpha{}_\beta \tau^i{}_\alpha(t,s)\tau_k{}^\beta(t,s)$$

with $(\tau_i{}^k)$ the inverse-transposed of $(\tau^i{}_k)$.

The **covariant derivative** ∇_X (X: vector field, R: tensor field) associated to τ is

$$(\nabla_X T)_p = \frac{d}{dt}\tau(0,t)T_{\gamma(t)}\Big|_{t=0}\ , \tag{2.7}$$

where $\gamma(t)$ is any curve through $p=\gamma(0)$ with $\dot\gamma(0)=X_p$.

Properties

(a) ∇_X is a linear map from tensor fields to tensor fields of the same type
(b) $\nabla_X f = Xf$
(c) $\nabla_X(\operatorname{tr} T) = \operatorname{tr}(\nabla_X T)$, ($\operatorname{tr}$ = any trace)
(d) $\nabla_X(T\otimes S) = \nabla_X T\otimes S + T\otimes\nabla_X S$

They follow from the corresponding properties of $\tau(t,s)$. For a 1-form ω we have

$$
\begin{aligned}
(\nabla_X\omega)(Y) &= \operatorname{tr}(\nabla_X\omega\otimes Y) = \operatorname{tr}\nabla_X(\omega\otimes Y) - \operatorname{tr}(\omega\otimes\nabla_X Y)\\
&= \nabla_X \operatorname{tr}(\omega\otimes Y) - \omega(\nabla_X Y) = X\omega(Y) - \omega(\nabla_X Y)\ .
\end{aligned} \tag{2.8}
$$

We write the general differentiation rule for a tensor field of type $\binom{1}{1}$

$$(\nabla_X T)(\omega, Y) = XT(\omega,Y) - T(\nabla_X\omega, Y) - T(\omega,\nabla_X Y)\ . \tag{2.9}$$

This is obvious from (2.8, 2.9) that due to (a–d) the operation ∇_X is completely determined by its action on vector fields Y. The latter is called an **affine connection**:

(i) $\nabla_X Y$ is a vector field depending linearly on X, Y
(ii) $\nabla_X Y$ is f-linear in X:

$$\nabla_{fX}Y = f\nabla_X Y\ , \qquad (f\in\mathcal{F})\ . \tag{2.10}$$

(iii) $\nabla_X(fY) = f\nabla_X Y + (Xf)Y$

Proof: (iii) is a special case of (d); (ii) is verified by means of its representation in a chart

$$(\nabla_X Y)^i = \frac{d}{dt}\tau^i{}_k(0,t)Y^k(x^1 + tX^1 + O(t^2), \ldots)\Big|_{t=0} = (Y^i{}_{,l} + \Gamma^i{}_{lk}Y^k)X^l \tag{2.11}$$

where we used (2.4). □

Conversely any affine connection entails a parallel transport (bijectively): In any chart with coordinate basis $(e_i, \ldots e_n)$ we have

$$\begin{aligned} \nabla_X Y &= \nabla_X(Y^i e_i) = (XY^i)e_i + Y^k(\nabla_X e_k) \\ &= Y^i{}_{,l}X^l e_i + Y^k X^l \nabla_{e_l} e_k \end{aligned}$$

which, after defining

$$\Gamma^i{}_{lk}(x) = \langle e^i, \nabla_{e_l} e_k \rangle \tag{2.12}$$

agrees with (2.11). One can show that (2.12) transforms according to (2.5), and hence defines a parallel transport.

The covariant derivative ∇

Example: By (2.9) $(\nabla_X T)(\omega, Y)$ is f-linear in all 3 variables ω, Y, X, and this defines a tensor field of type $\binom{1}{2}$ through

$$(\nabla T)(\omega, Y, X) = (\nabla_X T)(\omega, Y) \ .$$

The notation

$$T^i{}_{k;l} \equiv (\nabla T)^i{}_{kl}$$

for its components is customary, but a bit dangerous: for fixed i, k $T^i{}_{k;l}$ is not determined by the sole component $T^i{}_k(x)$! Examples:

$$\begin{aligned} Y^i{}_{;k} &= Y^i{}_{,k} + \Gamma^i{}_{kl}Y^l \ , \\ \omega_{i;k} &= \omega_{i,k} - \omega_l \Gamma^l{}_{ki} \ , \\ T^i{}_{k;r} &= T^i{}_{k,r} + \Gamma^i{}_{rl}T^l{}_k - \Gamma^l{}_{rk}T^i{}_l \ . \end{aligned}$$

2. Torsion and curvature

Let an affine connection be given on M, let X, Y, Z be vector fields. **Definitions:**

$$\begin{aligned} T(X,Y) &= \nabla_X Y - \nabla_Y X - [X,Y] \ , \\ R(X,Y) &= \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]} \ . \end{aligned}$$

To begin with, the **torsion** $T(X,Y)$ is a vector field and the **curvature** $R(X,Y)$ a linear map from tensor fields to tensor fields of the same type. They have however tensorial **character**:

- $T(X,Y)$ is antisymmetric and f-linear in X, Y and thus defines a tensor of type $\binom{1}{2}$ through

$$(\omega, X, Y) \mapsto \langle \omega, T(X,Y) \rangle \ .$$

- $R(X,Y)$ is antisymmetric in X, Y. The vector field $R(X,Y)Z$ is f-linear in X, Y, Z. Therefore R determines a tensor of type $\binom{1}{3}$ (**Riemann-Tensor**):

$$(\omega, Z, X, Y) \mapsto \langle \omega, R(X,Y)Z \rangle \equiv R^i{}_{jkl} \omega_i Z^j X^k Y^l \ .$$

Proof: We have

$$[fX, Y] = f[X,Y] - (Yf)X \ .$$

Thus

$$\begin{aligned} T(fX,Y) &= f\nabla_X Y - f\nabla_Y X - \underline{(Yf)X} - f[X,Y] + \underline{(Yf)X} = fT(X,Y) \ , \\ R(fX,Y) &= f\nabla_X\nabla_Y \underbrace{-\nabla_Y f \nabla_X}_{-f\nabla_Y\nabla_X - \underline{(Yf)\nabla_X}} - f\nabla_{[X,Y]} + \underline{(Yf)\nabla_X} = fR(X,Y) \end{aligned}$$

with cancellation of the underlined terms. The f-linearity in Z of $R(X,Y)Z$ follows from (d) of the next proposition. □

Proposition:

(a) $R(X,Y)f = 0$
(b) $R(X,Y)(S \otimes T) = (R(X,Y)S) \otimes T + S \otimes (R(X,Y)T)$
(c) $\operatorname{tr} R(X,Y)T = R(X,Y) \operatorname{tr} T$, (tr without contraction involving X or Y)
(d)

$$\langle \omega, R(X,Y)Z \rangle = -\langle R(X,Y)\omega, Z \rangle \tag{2.13}$$

Proof: (a) $R(X,Y)f = X(Yf) - Y(Xf) - [X,Y]f = 0$; (b) follows from the product rule for ∇_X (property (d)); (c) from (c) for ∇_X; (d) From (a–c) we have

$$\begin{aligned} 0 &= R(X,Y)\langle \omega, Z \rangle = R(X,Y) \operatorname{tr}(Z \otimes \omega) = \operatorname{tr} R(X,Y)(Z \otimes \omega) \\ &= \operatorname{tr}(R(X,Y)Z \otimes \omega) + \operatorname{tr}(Z \otimes R(X,Y)\omega) = \langle \omega, R(X,Y)Z \rangle + \langle R(X,Y)\omega, Z \rangle \ . \end{aligned}$$

□

Components (w.r.t. a coordinate basis $e_i = \partial/\partial x^i$, $e^i = dx^i$). From $[e_i, e_j] = 0$ we have

$$T^k{}_{ij} = \langle e^k, \nabla_{e_i} e_j - \nabla_{e_j} e_i \rangle = \Gamma^k{}_{ij} - \Gamma^k{}_{ji} \ . \tag{2.14}$$

In particular we have

$$\begin{aligned} & T = 0 \Longleftrightarrow \Gamma^k{}_{ij} = \Gamma^k{}_{ji} \ , \\ R^i{}_{jkl} &= \langle e^i, (\nabla_{e_k}\nabla_{e_l} - \nabla_{e_l}\nabla_{e_k}) e_j \rangle = \langle e^i, \nabla_{e_k}(\Gamma^s{}_{lj} e_s) - \nabla_{e_l}(\Gamma^s{}_{kj} e_s) \rangle \\ &= \Gamma^i{}_{lj,k} - \Gamma^i{}_{kj,l} + \Gamma^s{}_{lj}\Gamma^i{}_{ks} - \Gamma^s{}_{kj}\Gamma^i{}_{ls} \ . \end{aligned} \tag{2.15}$$

Bianchi identities for the special case **torsion** = 0:

$$\begin{aligned} &1) & R(X,Y)Z + \text{cycl.} &= 0 \ , \\ &2) & (\nabla_X R)(Y,Z) + \text{cycl.} &= 0 \ . \end{aligned} \tag{2.16}$$

Proof: 1) Let us write $X_1 = X$, $X_2 = Y$, $X_3 = Z$ and suppress the sum over $i = 1, 2, 3$ from the notation:

$$
\begin{array}{rcl}
R(X_i, X_{i+1})X_{i+2} & = & \underbrace{\nabla_{X_i}\nabla_{X_{i+1}}X_{i+2}} - \underbrace{\nabla_{X_{i+1}}\nabla_{X_i}X_{1+2}} - \nabla_{[X_i,X_{i+1}]}X_{1+2} \\
\text{cyclic permutation} & & \underbrace{\nabla_{X_{i+2}}\nabla_{X_j}X_{i+1} \quad \nabla_{X_{i+2}}\nabla_{X_{i+1}}X_i} \\
T = 0: & & \nabla_{X_{i+2}}[X_i, X_{i+1}]
\end{array}
$$

hence, because of (1.2), $R(X_i, X_{i+1})X_{i+2} = [X_{i+2}, [X_i, X_{i+1}]] = 0$.

2)

$$
\begin{array}{rr|r}
(\nabla_{X_i}R)(X_{i+1}, X_{i+2}) = \quad \nabla_{X_i}R(X_{i+1}, X_{i+2}) - R(X_{i+1}, X_{i+2})\nabla_{X_i} & & I \\
-R(\nabla_{X_i}X_{i+1}, X_{i+2}) - R(X_{i+1}, \nabla_{X_i}X_{i+2})\ , & & II
\end{array}
$$

where, through cyclic permutation,

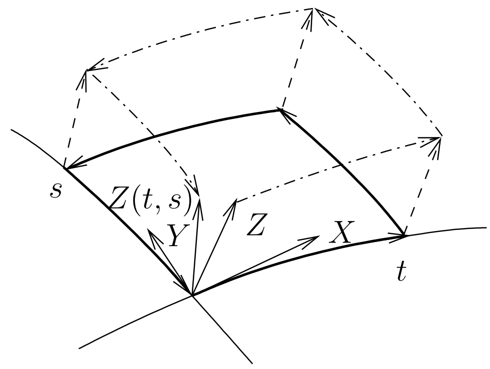
$$
\begin{aligned}
I &= \underline{\nabla_{X_i}\nabla_{X_{i+1}}\nabla_{X_{i+2}}} - \underline{\underline{\nabla_{X_i}\nabla_{X_{i+2}}\nabla_{X_{i+1}}}} - \nabla_{X_i}\nabla_{[X_{i+1},X_{i+2}]} \\
&\quad -\underline{\nabla_{X_{i+1}}\nabla_{X_{i+2}}\nabla_{X_i}} + \underline{\underline{\nabla_{X_{i+2}}\nabla_{X_{i+1}}\nabla_{X_i}}} + \nabla_{[X_{i+1},X_{i+2}]}\nabla_{X_i} \\
&= R([X_{i+1}, X_{i+2}], X_i) + \underbrace{\nabla_{[[X_{i+1},X_{i+2}],X_i]}}_{=0}\ , \\
II &= -R(\nabla_{X_{i+1}}X_{i+2}, X_i) + R(\nabla_{X_i}X_{i+2}, X_{i+1}) \\
&= -R(\nabla_{X_{i+1}}X_{i+2}, X_i) + R(\nabla_{X_{i+2}}X_{i+2}, X_i) = -R([X_{i+1}, X_{i+2}], X_i)\ .
\end{aligned}
$$

$\square$

In component notation:

$$
\begin{array}{lrcl}
1) & R^i{}_{jkl} + \text{cycl. } (jkl) & = & 0\ , \\
2) & R^i{}_{jkl;m} + \text{cycl. } (klm) & = & 0\ .
\end{array}
$$

On the meaning of curvature



Let X, Y be vector fields with corresponding flows φ_t, ψ_s satisfying $[X, Y] = 0$. Then $R(X, Y) = \nabla_X\nabla_Y - \nabla_Y\nabla_X$ and $\varphi_t \circ \psi_s = \psi_s \circ \varphi_t$, see (1.16). Let $\tau_X(t) : T_p \to T_{\varphi_t(p)}$ be the parallel transport along the orbit $\varphi_{t'}(p)$, $(0 \le t' \le t)$, of X, and similarly for $\tau_Y(s)$. By (2.7) we have $(d/dt)\tau_X(t)Z|_{t=0} = -\nabla_X Z$ for a vector field Z. We transport Z along a small loop consisting of orbits and obtain

$$
Z(t, s) := \tau_Y(-s)\tau_X(-t)\tau_Y(s)\tau_X(t)Z\ .
$$

Since $Z(t,s) = Z$ for $t = 0$ or $s = 0$, the lowest order term of the Taylor expansion $Z(t,s) - Z$ is proportional to ts. With

$$\begin{aligned}
\frac{\partial}{\partial t} Z(t,s)\Big|_{t=0} &= \tau_Y(-s)\nabla_X \tau_Y(s) Z - \nabla_X Z \ , \\
\frac{\partial^2}{\partial s \partial t} Z(t,s)\Big|_{t=s=0} &= (\nabla_Y \nabla_X - \nabla_X \nabla_Y) Z = -R(X,Y)Z \ .
\end{aligned}$$

we find

$$Z(t,s) = Z - tsR(X,Y)Z + O(|(t,s)|^3) \ :$$

The curvature measures the deviation of a vector, before and after the transport around the loop.

3. The Cartan structure equations

Let $(e_1, \dots e_n)$, $(e^1, \dots e^n)$ be any pair of dual bases of (local) vector fields, resp. 1-forms. For a given connection ∇ we define the **connection forms** ${\omega^i}_k$ by

$$ {\omega^i}_k(X) = \langle e^i, \nabla_X e_k \rangle \ , \tag{2.17}$$

resp. $\nabla_X e_k = {\omega^i}_k(X) e_i$. The ${\omega^i}_k$ are 1-forms because of (2.10). Conversely, any set of 1-forms ${\omega^i}_k$ defines a connection through

$$\nabla_X Y = \nabla_X (Y^k e_k) = \big[\underbrace{XY^i + Y^k {\omega^i}_k(X)}_{(\nabla_X Y)^i} \big] e_i \ . \tag{2.18}$$

From $\nabla_X \langle e^i, e_k \rangle = \nabla_X {\delta^i}_k = 0$ we have

$$\langle \nabla_X e^i, e_k \rangle = -{\omega^i}_k(X) \ .$$

These equations allow to express the components w.r.t that basis of the covariant derivative of any tensor field, e.g. of a 1-form Ω

$$(\nabla_X \Omega)_i = X\Omega_i - {\omega^i}_k(X)\Omega_k \ .$$

Remarks: 1) To the pair of bases $\bar{e}_i = {\phi_i}^k e_k$, $\bar{e}^i = {\phi^i}_k e^k$ there corresponds the connection forms

$$\bar{\omega}^i_k = {\phi_i}^l {\phi_k}^r {\omega^l}_r + {\phi_i}^l d{\phi_k}^l \ .$$

2) In a coordinate basis we have (cf. (2.12))

$${\omega^i}_k(e_l) = {\Gamma^i}_{lk} \ , \tag{2.19}$$

hence

$${\omega^i}_k(X) = {\Gamma^i}_{lk} X^l \ , \qquad \text{d.h.} \quad {\omega^i}_k = {\Gamma^i}_{lk} dx^l \ .$$

Definition

$$\begin{aligned}
T^i(X,Y) &= \langle e^i, T(X,Y) \rangle \ , & \qquad &(\textbf{Torsion forms}) \ , \\
{\Omega^i}_k(X,Y) &= \langle e^i, R(X,Y) e_k \rangle \ , & \qquad &(\textbf{Curvature forms}) \ .
\end{aligned}$$

These 2-forms are determined by the connection forms:

Cartan structure equation

$$\begin{aligned} T^i = & \quad de^i + \omega^i{}_k \wedge e^k \ , \\ \Omega^i{}_k = & \ d\omega^i{}_k + \omega^i{}_l \wedge \omega^l{}_k \ . \end{aligned} \tag{2.20}$$

Proof: From (1.24) we have

$$de^i(X,Y) = Xe^i(Y) - Ye^i(X) - e^i([X,Y]) \ ,$$

whereas (2.18), i.e.,

$$e^i(\nabla_X Y) = Xe^i(Y) + \omega^i{}_k \otimes e^k(X,Y) \ ,$$

implies

$$T^i(X,Y) = (\omega^i{}_k \wedge e^k)(X,Y) + \underbrace{Xe^i(Y) - Ye^i(X) - e^i([X,Y])}_{de^i(X,Y)}$$

since $\omega_1 \wedge \omega_2 = \omega_1 \otimes \omega_2 - \omega_2 \otimes \omega_1$ for 1-forms (cf. (1.18)). The 2nd structure equations follows similarly from (2.17), i.e.,

$$\nabla_Y e_k = \omega^l{}_k(Y) e_l \ ,$$

and from (2.18), giving

$$e^i(\nabla_X \nabla_Y e_k) = X\omega^i{}_k(Y) + \omega^i{}_l(X)\omega^l{}_k(Y)$$

and hence

$$\begin{aligned} \Omega^i{}_k(X,Y) &= e^i((\nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]})e_k) \\ &= (\omega^i{}_l \wedge \omega^l{}_k)(X,Y) + \underbrace{X\omega^i{}_k(Y) - Y\omega^i{}_k(X) - \omega^i{}_k([X,Y])}_{d\omega^i{}_k(X,Y)} \ . \end{aligned}$$

□

Components

$$T^i{}_{jk} = T^i(e_j, e_k) \ ; \qquad R^i{}_{jkl} = \Omega^i{}_j(e_k, e_l) \ , \tag{2.21}$$

resp.

$$T^i = \frac{1}{2} T^i{}_{jk} e^j \wedge e^k \ ; \qquad \Omega^i{}_j = \frac{1}{2} R^i{}_{jkl} e^k \wedge e^l \ .$$

Remark: In a coordinate basis (i.e., $e^i = dx^i$, $de^i = 0$, eqs. (2.21, 2.20, 2.19) allow to recover (eq5.2, 2.15).

Finally we write once more the **Bianchi identities**, again for the case of **torsion = 0**, but this time in the Cartan formalism

1) $$\Omega^i{}_k \wedge e^k = 0 \ ,$$
2) $$d\Omega^i{}_k = \Omega^i{}_l \wedge \omega^l{}_k - \omega^i{}_l \wedge \Omega^l{}_k \ .$$

Proof: 1) The exterior derivative of the first eq. (2.20) yields, because of $T^i = 0$,

$$0 = d(\omega^i{}_k \wedge e^k) = \underbrace{d\omega^i{}_k}_{(2.20):\ \Omega^i{}_k - \omega^i{}_l \wedge \omega^l{}_k} \wedge e^k - \omega^i{}_k \wedge \underbrace{de^k}_{-\omega^k_l \wedge e^l}$$

hence

$$\Omega^i{}_k \wedge e^k = \omega^i{}_l \wedge \omega^l{}_k \wedge e^k - \omega^i{}_k \wedge \omega^l{}_k \wedge e^l = 0 \ .$$

2) The exterior derivative of the second eq. (2.20) yields

$$d\Omega^i{}_k = \underbrace{d\omega^i{}_l}_{\Omega^i{}_l - \omega^i{}_j \wedge \omega^j{}_k} \wedge \omega^l{}_k - \omega^i{}_l \wedge \underbrace{d\omega^l{}_k}_{\Omega^l{}_k - \omega^l{}_j \wedge \omega^j{}_k} = \Omega^i{}_l \wedge \omega^l{}_k - \omega^i{}_l \wedge \Omega^l{}_k \ .$$

□

One checks, e.g. by using a coordinate basis, that the above form of the Bianchi identities agrees with the one seen previously.

3. Pseudo-Riemannian manifolds

1. Metric

Let M be equipped with a **pseudo-Riemannian metric**: a symmetric, non-degenerate tensor field

$$g(X,Y) \equiv (X,Y)$$

of type $\binom{0}{2}$. Non-degenerate means, that for any $p \in M$ and $(X, Y \in T_p)$ one has

$$g_p(X,Y) = 0\ , \quad \forall Y \in T_p \ \Rightarrow\ X = 0\ .$$

In components:

$$(X,Y) = g_{ik}X^iY^k$$

with $g_{ik} = g_{ki}$ and $\det(g_{ik}) \neq 0$. Positivity (and hence a Riemannian metric) will not be assumed.

The metric allows to identify vector fields with 1-forms:

$$X \mapsto gX\ , \quad \omega \mapsto g^{-1}\omega \tag{3.1}$$

by means of

$$\langle gX, Y\rangle = (X,Y)\ , \quad (g^{-1}\omega, Y) = \langle \omega, Y\rangle\ .$$

The maps (3.1) are called **lowering**, resp. **raising indices**, because for $\tilde{X} = gX$, $\tilde{\omega} = g^{-1}\omega$ we have

$$\tilde{X}_i = g_{ik}X^k\ , \quad \tilde{\omega}^i = g^{ik}\omega_k\ ,$$

where (g^{ik}) denotes the inverse of the matrix (g_{ik}). By the same token we can identify different types of tensor fields having the same number of indices. In components (e.g.):

$$T^i{}_k = T_{lk}g^{il} = T^{il}g_{lk}\ .$$

Given a basis $(e_1, \dots e_n)$ of T_p, the covectors of the dual basis $(e^1, \dots e^n)$ become themselves vectors; indeed, $e_i = g_{ij}e^j$.

2. The Riemann connection

The metric distinguishes an affine connection, called **Riemann** (or **Levi-Civita**) connection.

Theorem: There is a unique connection with vanishing torsion and

$$\nabla g = 0\ . \tag{3.2}$$

It is given by

$$2(\nabla_X Y, Z) = X(Y,Z) + Y(Z,X) - Z(X,Y) - ([Y,Z]X) + ([Z,X],Y) + ([X,Y],Z)\ . \tag{3.3}$$

Proof: uniqueness: because of (3.2) we have

$$\begin{aligned} 0 &= \nabla g(X_i, X_{i+1}, X_{i+2}) = (\nabla_{X_{i+2}}g)(X_i, X_{i+1}) \\ &= X_{i+2}g(X_i, X_{i+1}) - g(\nabla_{X_{i+2}}X_i, X_{i+1}) - \underbrace{g(X_i, \nabla_{X_{i+2}}X_{i+1})}_{g(\nabla_{X_{i+2}}X_{i+1}, X_i)} \end{aligned} \tag{3.4}$$

By taking the combination $(3.4)_{i+1} + (3.4)_{i+2} - (3.4)_i$, we obtain

$$\begin{aligned}0 = X_i g(X_{i+1}, X_{i+2}) + X_{i+1} g(X_{i+2} X_i) - X_{i+2} g(X_i, X_{i+1})\\ - g(\underbrace{\nabla_{X_{i+1}} X_{i+2} - \nabla_{X_{i+2}} X_{i+1}}_{[X_{i+1},X_{i+2}]}, X_i) + g(\underbrace{\nabla_{X_{i+2}} X_i - \nabla_{X_i} X_{i+2}}_{[X_{i+2},X_i]}, X_{i+1})\\ - g(\underbrace{\nabla_{X_i} X_{i+1} + \nabla_{X_{i+1}} X_i}_{2\nabla_{X_i} X_{i+1} - [X_i, X_{i+1}]}, X_{i+2}) \ ,\end{aligned} \tag{3.5}$$

(underbracing uses torsion = 0), which for $X_1 = X$, $Y_2 = Y$, $X_3 = Z$ agrees with (3.3). That determines $\nabla_X Y$ since g is non-degenerate.

Existence: By (3.3) a vector field $\nabla_X Y$ is defined. One verifies that it enjoys the properties of a connection, e.g. the f-linearity in X:

$$\begin{aligned} 2(\nabla_{fX} Y, Z) &= fX(Y,Z) + \underbrace{Y(fX,Z)}_{fY(X,Z)+(Yf)(X,Z)} - \underbrace{Z(fX,Y)}_{fZ(X,Y)+(Zf)(X,Y)} \\ &\quad -([Y,Z], fX) + (\underbrace{[Z, fX]}_{f[Z,X]+(Zf)X}, Y) + (\underbrace{[fX, Y]}_{f[X,Y]-(Yf)X}, Z) \\ &= 2f(\nabla_X Y, Z) \ , \end{aligned}$$

i.e. $\nabla_{fX} Y = f \nabla_X Y$. The vanishing of the torsion is manifest from

$$2(\nabla_X Y - \nabla_Y X, Z) = 2([X,Y], Z) \ .$$

Finally (3.3), or its equivalent form (3.5), implies, by taking $(3.5)_{i+1} + (3.5)_{i+2}$, the equation $(3.4)_i$, which is in turn equivalent to (3.2). $\square$

In a chart the Riemann connection reads

$$\Gamma^i{}_{lk} = \frac{1}{2} g^{ij} (g_{lj,k} + g_{kj,l} - g_{lk,j}) \ , \tag{3.6}$$

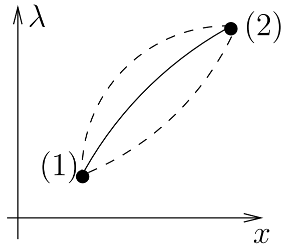
since for $X = \partial/\partial x^l$, $Y = \partial/\partial x^k$, $Z = \partial/\partial x^j = g_{ij} dx^i$ (3.3) reads (cf. (2.12))

$$2 g_{ij} \Gamma^i{}_{lk} = g_{kj,l} + g_{jl,k} - g_{lk,j} \ .$$

Geodesics:

We define geodesics $x(\lambda)$ by the variational principle

$$\delta \int_{(1)}^{(2)} d\lambda \, (\dot{x}, \dot{x}) = 0$$



with fixed endpoints.

(Here $\dot{x} = dx/d\lambda$ denotes the tangent vector). In any chart the geodesics satisfy the Euler-Lagrange equations corresponding to the Lagrangian

$$L(x, \dot{x}) = \frac{1}{2} g_{lk}(x) \dot{x}^l \dot{x}^k \ , \tag{3.7}$$

namely:

$$
\begin{aligned}
0 = \frac{d}{d\lambda}\frac{\partial L}{\partial \dot{x}^j} - \frac{\partial L}{\partial x^j} &= \frac{d}{d\lambda}(g_{lj}\dot{x}^l) - \frac{1}{2}g_{lk,j}\dot{x}^l\dot{x}^k \\
&= \underbrace{g_{lj,k}\dot{x}^l\dot{x}^k}_{(1/2)(g_{lj,k}+g_{kj,l})\dot{x}^l\dot{x}^k} + g_{ij}\ddot{x}^i - \frac{1}{2}g_{lk,j}\dot{x}^l\dot{x}^k
\end{aligned}
$$

i.e.

$$
g_{ij}\ddot{x}^i + \frac{1}{2}(g_{lj,k} + g_{kj,l} - g_{lk,j})\dot{x}^l\dot{x}^k = 0 ,
$$

or

$$
\ddot{x}^i + \Gamma^i{}_{lk}\dot{x}^l\dot{x}^k = 0 \tag{3.8}
$$

(**Geodesic equation**). It states that the vector $\dot{x}$ is parallel transported along the geodesic (cf. (2.3)).

Moreover, (3.8) is invariant under reparameterization $\lambda \mapsto \lambda'$ only if $d^2\lambda'/d\lambda^2 = 0$. The parameterization is thus fixed by (3.8) up to $\lambda \mapsto a\lambda + b$ (with a, b constants): λ is then called an **affine parameter**.

Properties of the Riemann connection

(a) The inner product of any two vectors remains constant upon parallel transporting them along any curve γ:

$$
(X(t), Y(t))_{\gamma(t)} = (X, Y)_{\gamma(0)} \tag{3.9}
$$

with $X(t) = \tau(t,0)X$, $Y(t) = \tau(t,0)Y$ and X, $Y \in T_{\gamma(0)}$. Indeed, because of $\nabla g = 0$ we have $g_{\gamma(t)} = \tau(t,0)g_{\gamma(0)}$, so that (3.9) is equivalent to

$$
(\tau(t,0)g_{\gamma(0)})(\tau(t,0)X, \tau(t,0)Y) = g_{\gamma(0)}(X, Y) ,
$$

which holds true by (2.6).

(b) The covariant derivative commutes with raising and lowering indices, e.g.

$$
T^i{}_{k;l} = (g_{km}T^{im})_{;l} = g_{km}T^{im}{}_{;l}
$$

because $g_{km;l} = 0$. The same without reference to coordinates:

$$
\nabla_X \circ g = g \circ \nabla_X , \tag{3.10}
$$

where g denotes the map (3.1). **Proof:** By (2.8, 3.4) we have

$$
\langle \nabla_X gY, Z\rangle = X\langle gY, Z\rangle - \langle gY, \nabla_X Z\rangle = \langle g\nabla_X Y, Z\rangle
$$

for arbitrary vector fields Y, Z.

(c) **Riemann tensor**

The following symmetries apply:

$$
\begin{aligned}
(W, R(X,Y)Z) &= -(Z, R(X,Y)W) , && (3.11)\\
(W, R(X,Y)Z) &= (X, R(W,Z)Y) . && (3.12)
\end{aligned}
$$

Proof: From (3.10) we have $R(X,Y)g = gR(X,Y)$ and, together with (2.13), also (3.11). Because of the 1st Bianchi identity (2.16) the l.h.s. of (3.12) equals

$$-(W, R(Y,Z)X) - (W, R(Z,X)Y)$$

as well as, in view of (3.11),

$$(Z, R(Y,W)X) + (Z, R(W,X)Y) \ .$$

The sum of the two expressions is symmetric in $(X,Y) \leftrightarrow (W,Z)$. $\square$

We summarize all symmetries of the Riemann tensor:

$R^i{}_{jkl} = -R^i{}_{jlk}$		**always**
$\sum_{(jkl)} R^i{}_{jkl} = 0$	1. Bianchi id.	**vanishing torsion**
$\sum_{(klm)} R^i{}_{jkl;m} = 0$	2. Bianchi id.	
$R_{ijkl} = -R_{jikl}$		**Riemann connection**
$R_{ijkl} = R_{klij}$		

Here $\sum_{(jkl)}$ means the sum over the cyclic permutations of j,k,l.

(d) **Ricci and Einstein tensors**

Definition:

$$R_{ik} = R^j{}_{ijk} \qquad (\textbf{Ricci tensor})$$

$$R = R^i{}_i \qquad (\textbf{scalar curvature})$$

$$G_{ik} = R_{ik} - \frac{1}{2}Rg_{ik} \qquad (\textbf{Einstein tensor})$$

We have:

$$R_{ik} = R_{ki} \ , \qquad G_{ik} = G_{ki}$$

$$\left.\begin{array}{l} R_i{}^k{}_{;k} = \frac{1}{2}R_{;i} \\ G_i{}^k{}_{;k} = 0 \end{array}\right\} \qquad \text{(contracted 2nd Bianchi identity)} \tag{3.13}$$

Proof: $R_{ik} = g^{jl}R_{lijk} = g^{il}R_{jkli} = R_{ki}$.

2nd Bianchi identity:

$$R^i{}_{jkl;m} + R^i{}_{jlm;k} + R^i{}_{jmk;l} = 0 \ .$$

(ik)-trace:

$$R_{jl;m} + \underbrace{R^i{}_{jlm;i}}_{-g^{ik}R_{jklm;i}} - R_{jm;l} = 0 \ ,$$

$$R^j{}_{l;m} - g^{ik}R^j{}_{klm;i} - R^j{}_{m;l} = 0 \ ,$$

(jm)-trace:

$$\underbrace{R^j{}_{l;j} + g^{ik}R_{kl;i}}_{2R^j{}_{l;j}} - R_{;l} = 0 \ .$$

$\square$

3. Supplementary material

Normal coordinates

The signature of the metric g_p is the same for all $p \in M$ (if M is connected). Let

$$\eta_{ij} = \begin{cases} 0\,, & (i \neq j) \\ \pm 1\,, & (i = j) \end{cases}$$

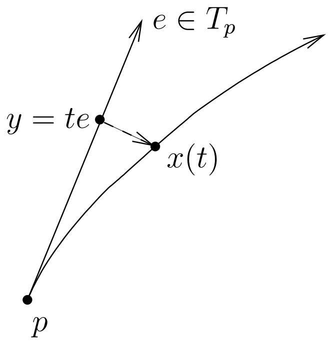
be its normal form.

Theorem: In some neighborhood of any point $p \in M$ there is a chart such that $x^i = 0$ at p and

$$\begin{aligned} g_{ij}(0) &= \eta_{ij}\,, \\ g_{ij,l}(0) &= 0\,, \qquad \text{i.e.} \quad {\Gamma^i}_{lj}(0) = 0\,. \end{aligned} \tag{3.14}$$

Proof: We first pick local coordinates x^i near p such that $x^i = 0$ at p and $g_{ij}\ldots(0) = \eta_{ij}$, where the latter condition can be achieved by means of a linear transformation. Then we construct the **exponential map** from $T_p(M)$ to M:

Let $e \in T_p$. The curve $t \mapsto x(t)$ is the solution of the geodesic equation (3.8) with $\dot{x}(0) = e$. The map $\exp : y = te \mapsto x(t)$ is uniquely defined, i.e. independent of the factorization $y = te$. Thereby a neighborhood of the origin in $T_p(M)$ is mapped differentiably to M. By the geodesic equation we then have



$$\begin{aligned} x^i(t) &= t\dot{x}^i(0) + \frac{1}{2}t^2\ddot{x}^i(0) + O(t^3) \\ &= y^i - \frac{1}{2}{\Gamma^i}_{lk}(0)y^l y^k + O(y^3)\,, \end{aligned}$$

and in particular $\partial x^i/\partial y^j = {\delta^i}_j$ at $y = 0$. Hence exp is a local diffeomorphism and we can take the y^i as new local coordinates. Since the geodesics through $y = 0$ then become straight lines, we have in the new coordinates

$${\Gamma^i}_{lk}(te)e^l e^k = 0$$

for all $e \in T_p$. Because of the symmetry ${\Gamma^i}_{lk} = {\Gamma^i}_{kl}$ we have

$${\Gamma^i}_{lk}(0) = 0\,.$$

This is equivalent to $g_{ij,l}(0) = 0$, since then $0 = g_{ij;l} = g_{ij,l}$, while converse is evident from (3.6).

The volume element

The metric, first defined on vector fields and 1-forms generalizes to tensor fields of type $\binom{0}{p}$ by means of

$$(\omega_1 \otimes \ldots \otimes \omega_p, w_1 \otimes \ldots \otimes w_p)_p := \frac{1}{p!}\prod_{i=1}^{p}(\omega_i, w_i)$$

and bilinearity. It remains non-degenerate. In particular, it is defined on n-forms (with signature $\sigma = \pm 1$). On an orientable manifold there is an n-form η, unique up to the sign, with

$$(\eta, \eta)_n = \sigma \ .$$

η is called the volume form of the metric g. W.r.t. a basis of 1-forms we have

$$\eta = \pm |g|^{1/2} e^1 \wedge \ldots \wedge e^n \ ,$$

where

$$g = \det(g_{ij}) \ , \qquad g_{ij} = g(e_i, e_j) \ .$$

Indeed,

$$\begin{aligned} (\eta, \eta)_n &= |g| \left(e^1 \wedge \ldots \wedge e^n, e^1 \wedge \ldots \wedge e^n \right)_n = |g| \sum_{\pi \in S_n} \operatorname{sgn} \pi \prod_{i=1}^{n} \left(e^i, e^{\pi(i)} \right) \\ &= |g| \underbrace{\det(g^{ij})}_{g^{-1}} = \operatorname{sgn} g = \sigma \ . \end{aligned}$$

In components

$$\eta_{i_1 \ldots i_n} = \pm |g|^{1/2} \ \varepsilon_{i_1 \ldots i_n} \ ,$$

where

$$\varepsilon_{i_1 \ldots i_n} = \operatorname{sgn} \begin{pmatrix} 1 \ldots n \\ i_1 \ldots i_n \end{pmatrix} \ .$$

The structure equations of the Riemann connection

Theorem: In any basis (not necessarily a coordinate basis) the connection coefficients ${\omega^i}_k$, cf. (2.17), are uniquely determined by

$$\omega_{ik} + \omega_{ki} = dg_{ik} \ , \qquad (\nabla g = 0) \tag{3.15}$$

$$de^i + {\omega^i}_k \wedge e^k = 0 \ , \qquad (\text{torsion zero}) \tag{3.16}$$

where we set

$$\omega_{ik} = g_{il} {\omega^l}_k \ .$$

Proof: For all X, e_i, e_k one has

$$\begin{aligned} 0 &= (\nabla_X g)(e_i, e_k) = X \underbrace{g(e_i, e_k)}_{g_{ik}} - g(\underbrace{\nabla_X e_i}_{{\omega^l}_i(X) e_l}, e_k) - g(e_i, \underbrace{\nabla_X e_k}_{{\omega^l}_k(X) e_k}) \\ &= dg_{ik}(X) - {\omega^l}_i(X) g_{lk} - {\omega^l}_k(X) g_{ik} \ . \end{aligned}$$

Thus (3.15) is equivalent to $\nabla g = 0$. According to (2.19), eq. (3.16) means $T = 0$. Conversely, these two equations determine, by the theorem on p. 24 the connection (and hence the connection forms) uniquely. □

4. Time, space and gravitation

1. The classical relativity principle

Rigid frames are at the basis of the classical idea of time and space. Newtonian Mechanics distinguishes a special class of trajectories: those of **free particles**. The 1st Law postulates the existence of special rigid frames, so-called **inertial frames** (IF), in which **all** such trajectories take the simple form

$$\ddot{\vec{x}} = 0\,.$$

The **classical relativity principle** (or equivalence principle) postulates that the equations of motion of any isolated system read the same in all IF. The 2nd Law specifies the deviation from a free trajectory

$$m_i \ddot{\vec{x}}_i = \vec{F}_i\left(\vec{x}_1, \ldots, \vec{x}_N\right)\,,$$

where the inertial mass m_i is a property of the i-th particle, and $\vec{F}_i$ are given by force laws, such as

$$\vec{F} = e\vec{E}\,, \qquad (e: \text{ electric charge})$$

for a particle in an electric field $\vec{E}$, or

$$\vec{F} = \widetilde{m}\vec{g}\,, \qquad (\widetilde{m}: \text{ gravitational mass})$$

for a particle in a gravitational field $\vec{g}$. Remarkable and without explanation in the present context is the fact that

$$m = \widetilde{m}\,,$$

whence

$$\ddot{\vec{x}} = \vec{g} \tag{4.1}$$

for all freely falling particles.

2. The Einstein equivalence principle

(EP) Einstein interprets (4.1) in the sense that the "standard of motion" is not given by trajectories of free, but rather of freely falling particles. In this sense gravity is not a proper force, but appears as an inertial force, whose proportionality to m is obvious. A strengthening of this point of view is the EP (1911).

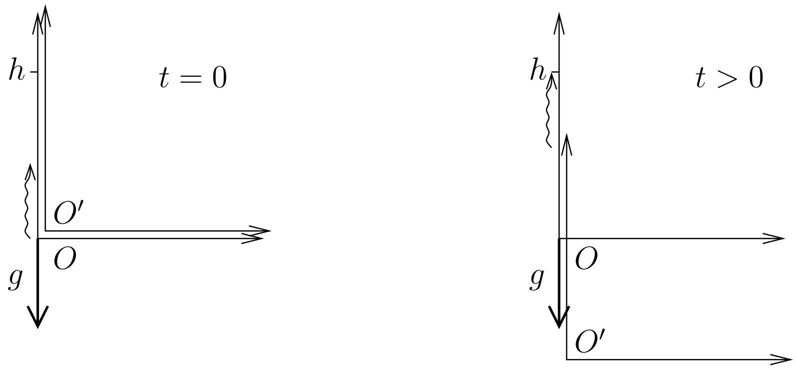
> "All freely falling, non-rotating local inertial frames (for short: local IF) are equivalent w.r.t. all local experiments therein."

Remarks: 1) A (local) reference frame is non-rotating, if freely falling particles do not experience any velocity-dependent (Coriolis-) acceleration, locally.

2) The above formulation of the EP is heuristic, because the notion of local experiment is vague. We stress that the relative deviation of nearby freely falling particles does not constitute a local experiment.

Application: The gravitational redshift

We take the classical idea of space and time for granted and consider two reference frames: O, where we have a homogeneous gravitational field $\vec{g}$, and O' which is in free fall. At time $t = 0$ the two coincide and are instantaneously at rest to one another.



At $t = 0$ and at $\vec{x} = \vec{x}\,' = 0$ light of frequency ν is emitted upwards. It reaches height h w.r.t. O after time $t = h/c$. According to the EP the frequency measured in O' is still ν. But since O' has then acquired the velocity O' relatively to O, the latter finds the Doppler shifted frequency

$$\bar{\nu} = \nu\Big(1 + \frac{v}{c}\Big) = \nu\left(1 - \frac{gh}{c^2}\right) . \tag{4.2}$$

Upon raising in the gravitational field the frequency decreases (or: it is shifted towards the red).

3. The postulates of general relativity (GR)

The postulates (Einstein 1915) clarify the EP:

1. Time and space form a 4-dimensional pseudo-Riemannian manifold M: Its points p represent events and the metric g of signature $(1, -1, -1, -1)$ describes measurements by means of (ideal) clocks and rods.

2. Physical laws are relations among tensors.

3. With the exception of the metric g physical laws only contain quantities already present in special relativity (SR).

4. In normal coordinates (see p. 28) about an event $p \in M$ (**local inertial frame**) the laws of SR hold true.

Remarks:

About 1: An ideal clock of world line $x = x(\lambda)$ measures (infinitesimally) the time $\Delta\tau$

$$c^2(\Delta\tau)^2 = g(\dot{x}, \dot{x})(\Delta\lambda)^2 .$$

An ideal (infinitesimal) rod is represented by the world line $x(\lambda)$ of one of its endpoints and by a vector $\Delta x(\lambda)$ with $g(\dot{x}, \Delta x) = 0$. Its length Δl is

$$(\Delta l)^2 = -g(\Delta x, \Delta x) .$$

In particular, if in some coordinates the world line of the clock is $x = (ct, 0, 0, 0)$, then

$$(\Delta\tau)^2 = g_{00}(x)(\Delta t)^2 . \tag{4.3}$$

One should thus distinguish between measurements by means of clocks and rods on one hand and coordinates of a chart on the other. They agree however locally in the neighborhood of an event, if they are represented as being at rest in the chart and the metric at the event is the **Minkowski metric** $\eta_{\mu\nu}$.

In principle it is to be decided on the basis of the physical laws whether a given clock or rod is ideal.

About 2: The physical laws read the same in all coordinates (provided the physical quantities are transformed suitably): **general covariance**.

About 4: Gravity can be transformed away locally.

Thanks to the above postulates the physical laws in presence of an external (i.e., given) gravitational field are essentially determined. The climax of GR are however the field equations of gravitation, which will be introduced in the next chapter.

4. Transition SR → GR

a) **Law of inertia**

SR		GR	
$\ddot{x}^\mu = 0\,,$		$\ddot{x}^\mu + \Gamma^\mu{}_{\nu\sigma}\dot{x}^\nu\dot{x}^\sigma = 0\,,$	(4.4)
$(\dot{x}, \dot{x}) = c^2\,,$	$\longrightarrow$	$(\dot{x}, \dot{x}) = c^2\,,$	(4.5)
"free particle"		"free falling particle"	

($\dot{} = d/d\tau$, τ: proper time). The equations on the right agree with those on the left in a local inertial frame, but are generally covariant. The geodesic equation (4.4) describes the effect of the "gravitational field" on an otherwise free particle: the r.h.s. in

$$\ddot{x}^\mu = -\Gamma^\mu{}_{\nu\sigma}\dot{x}^\nu\dot{x}^\sigma \tag{4.6}$$

can be viewed as gravitational force, hence actually the $\Gamma^\mu{}_{\nu\sigma}$ (not the $g_{\mu\nu}$) as components of the gravitational field. That one can be **transformed away** by (3.14) at any point of spacetime. The "equivalence of gravitational and inertial mass" is now automatic: the mass just does not appear.

Remark: Postulate 3 can be weakened, to the extent that one allows for a connection ∇, which is a priori independent of the metric. The existence of normal coordinates ($\Gamma^\mu{}_{\nu\sigma}(0) = 0$) requires that ∇ is torsion-free. Since there the laws of SR are presumed valid, we also have $g_{\mu\nu}(0) = \eta_{\mu\nu}$. That suffices in order to justify eqs. (4.4, 4.5). Their compatibility implies $\nabla g = 0$, cf. (3.9), i.e. ∇ is after all the Riemann connection.

b) For **light rays** we analogously have:

$$\begin{array}{ccc} \text{SR} & & \text{GR} \\ \ddot{x}^\mu = 0\,, & & \ddot{x}^\mu + \Gamma^\mu{}_{\nu\sigma}\dot{x}^\nu\dot{x}^\sigma = 0\,, \\ (\dot{x},\dot{x}) = 0\,, & \longrightarrow & (\dot{x},\dot{x}) = 0\,, \\ & & \text{(null geodesics)} \end{array} \tag{4.7}$$

Here (4.7) describes the **light deflection** in a gravitational field. Actually the full Maxwell theory can be formulated covariantly: it suffices to replace partial derivatives (of 1st order) by covariant ones. We view the electromagnetic field tensor F as an antisymmetric tensor field of type $\binom{0}{2}$. The homogeneous Maxwell equations then read

$$F_{\mu\nu,\sigma} + \text{cycl.} = 0 \longleftrightarrow F_{\mu\nu;\sigma} + \text{cycl.} = 0\,, \tag{4.8}$$

because the second form reduces to the first one in a local reference frame. The inhomogeneous equations read

$$F^{\mu\nu}_{;\mu} = \frac{1}{c}j^\nu\,. \tag{4.9}$$

Eq. (4.9) again implies charge conservation

$$j^\nu{}_{;\nu} = 0\,,$$

because by $F^{\mu\nu} = -F^{\nu\mu}$ we have

$$F^{\mu\nu}{}_{;\mu\nu} = \underbrace{F^{\mu\nu}{}_{;\nu\mu}}_{-F^{\nu\mu}{}_{;\nu\mu}} + \underbrace{\underbrace{R^\mu{}_{\tau\mu\nu}}_{R_{\tau\nu}}\ F^{\tau\nu} + \underbrace{R^\nu{}_{\tau\mu\nu}}_{-R_{\tau\mu}}\ F^{\mu\tau}}_{(R_{\tau\nu}-R_{\nu\tau})F^{\tau\nu}=0} = -F^{\mu\nu}{}_{;\mu\nu}\,.$$

The energy-momentum tensor is

$$T^{\mu\nu} = F^\mu{}_\sigma F^{\sigma\nu} - \frac{1}{4}F_{\rho\sigma}F^{\sigma\rho}g^{\mu\nu} \tag{4.10}$$

and for a “freely falling” field we have

$$T^{\mu\nu}{}_{;\nu} = 0\,.$$

The representation of the electromagnetic field in terms of the potentials is

$$F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu} = A_{\nu;\mu} - A_{\mu;\nu}\,.$$

c) The **equations of motion of a charged particles** (charge e, mass m) in an electromagnetic field and in presence of gravity now read

$$\ddot{x}^\mu + \Gamma^\mu{}_{\nu\sigma}\dot{x}^\nu\dot{x}^\sigma = \frac{e}{mc}F^{\mu\nu}\dot{x}_\nu\,, \tag{4.11}$$

because they are generally covariant (the l.h.s. is $\nabla_{\dot{x}}\dot{x}$, hence a vector) and reduce to the equations of SR in a local reference frame. Moreover, one verifies that (4.11) are the Euler-Lagrange equations corresponding to the manifestly covariant Hamilton principle

$$\delta\int_{(1)}^{(2)} d\tau\left(c^2 + \frac{e}{mc}(\dot{x},A)\right) = 0$$

with fixed endpoints $(1),(2)$ in M.

5. Transition geodesic equation → Newton's equation of motion

Newton's equation of motion appears on an approximation under certain assumptions. We use coordinates which in the immediate (infinitesimal) neighborhood of the observer have the meaning of lengths and times:

$$g_{\mu\nu} = \eta_{\mu\nu} \qquad \text{for } x = (ct, 0, 0, 0),$$

$$\eta_{\mu\nu} = \begin{pmatrix} 1 & & & \\ & -1 & & 0 \\ 0 & & -1 & \\ & & & -1 \end{pmatrix} .$$

We follow trajectories within a region where the gravitational field is weak in the sense that

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \ , \qquad |h_{\mu\nu}| \ll 1 \ . \tag{4.12}$$

In particular we have $h_{\mu\nu,0} = 0$ at the origin $\vec{x} = 0$. At first, let the particle be nearly at rest

$$\dot{x}^\mu = (c, \vec{0}) \ , \qquad (\dot{} = \frac{d}{d\tau} = \frac{d}{dt} \quad \text{up to } O(v^2) + O(h)) \ .$$

Then (4.4) reads

$$\ddot{x}^i = -c^2 \Gamma^i{}_{00} \ ,$$

where in linear approximation in h

$$\Gamma^i{}_{00} = \frac{1}{2}\eta^{ik}(h_{0k,0} + h_{0k,0} - h_{00,k}) = \frac{1}{2}h_{00,i} - h_{i0,0} = \frac{1}{2}h_{00,i} \ , \tag{4.13}$$

$(i = 1, 2, 3)$; in the last step we evaluated at $\vec{x} = 0$. Thus

$$\ddot{\vec{x}} = -\vec{\nabla}\varphi \ , \qquad \varphi = \frac{1}{2}c^2 h_{00} \ .$$

Put differently: In a weak gravitational field we have

$$g_{00} = 1 + \frac{2\varphi}{c^2} \ ; \qquad \varphi : \text{Newtonian potential.} \tag{4.14}$$

We now retain terms $\propto \vec{v}$ (i.e., we neglect only terms $O(v^2)$); then $\dot{x}^\mu = (c, \vec{v})$ and (4.4) becomes

$$\ddot{x}^i = -c^2 \Gamma^i{}_{00} - 2c\Gamma^i{}_{0j}\dot{x}^j \tag{4.15}$$

with

$$\Gamma^i{}_{0j} = \frac{1}{2}\eta^{ik}(h_{0k,j} + h_{jk,0} - h_{0j,k}) = \frac{1}{2}(h_{0j,i} - h_{0i,j}) \ .$$

Correspondingly we keep terms $O(\vec{x})$ in (4.13), since $\vec{x} \sim \vec{v}t$. For comparison, the classical equation of motion of a freely falling particle in an accelerated reference frame (not an IS) is

$$\ddot{\vec{x}} = -\vec{\nabla}\varphi - 2\vec{\omega} \wedge \dot{\vec{x}} - \vec{\omega} \wedge (\vec{\omega} \wedge \vec{x}) - \dot{\vec{\omega}} \wedge \vec{x} \ , \tag{4.16}$$

where the inertial acceleration is included in $\vec{\nabla}\varphi$. Now (4.15, 4.16) agree locally for

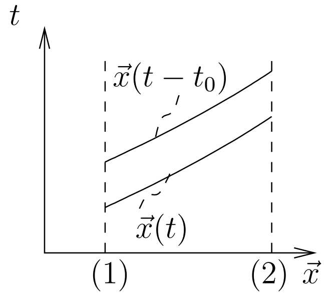
$$\begin{aligned} g_{00} &= 1 + \frac{2}{c^2}(\varphi + \frac{1}{2}(\vec{\omega} \wedge \vec{x})^2) \ . \\ g_{0i} &= -\frac{1}{c}(\vec{\omega} \wedge \vec{x})_i \ . \end{aligned}$$

This follows from $c(h_{0j,i}-h_{0i,j}) = \varepsilon_{jik}\omega_k$, $c\Gamma^i{}_{0j}\dot{x}^j = (\vec{\omega}\wedge\dot{\vec{x}})_i$, $\vec{\omega}\wedge(\vec{\omega}\wedge\vec{x}) = -(1/2)\vec{\nabla}(\vec{\omega}\wedge\vec{x})^2$ and $c^2 h_{i0,0} = (\dot{\vec{\omega}}\wedge\vec{x})_i$.

Redshift

We consider a metric which is independent of time in suitable coordinates $(ct, \vec{x})$:

$$g_{\mu\nu,0} = 0\ .$$



If $(t, \vec{x}(t))$, $(t_1 \le t \le t_2)$, is a (null-) geodesic, then so is $(t, \vec{x}(t-t_0))$, $(t_1+t_0 \le t \le t_2+t_0)$. In particular, the difference Δt between consecutive minima of a light wave is constant along the ray. The proper time τ of an observer resting at $\vec{x}$ is related to coordinate time according to (4.3)

$$(\Delta\tau)^2 = g_{00}(\vec{x})(\Delta t)^2\ .$$

Hence we have for the frequency ν at the positions (1), (2) of a light ray.

$$\frac{\nu_2}{\nu_1} = \frac{(\Delta\tau)_1}{(\Delta\tau)_2} = \sqrt{\frac{g_{00}(\vec{x}_1)}{g_{00}(\vec{x}_2)}}\ . \tag{4.17}$$

Remarks: 1) In the situation of (4.14) (and hence with $2\varphi \ll c^2$) we have

$$\frac{\nu_2}{\nu_1} = \sqrt{1 - 2\frac{\Delta\varphi}{c^2}} \approx 1 - \frac{\Delta\varphi}{c^2}$$

with $\Delta\varphi = \varphi|_1^2$. This agrees with (4.2) ($\Delta\varphi = gh$).

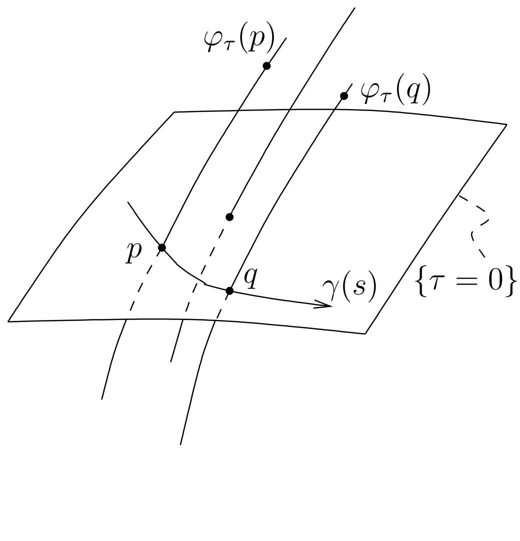
2) The EP is incompatible with SR, at least if its metric $\eta_{\mu\nu}$ is supposed to describe time measurements, see (4.3): With any light ray, a time translate thereof is one too (even, if it weren't a null geodesic). With $g_{\mu\nu} = \eta_{\mu\nu}$ we would always get $\nu_2/\nu_1 = 1$ (no redshift). Gravitation can thus not be accommodated within SR.

6. Geodesic deviation

Family of geodesics $x(\tau)$ with 4-velocity field u (cf. (4.4)):

$$\frac{dx}{d\tau} = u(x(\tau))\ , \qquad \nabla_u u = 0\ , \qquad g(u,u) = c^2\ .$$

Let φ_τ be the flow generated by u. We investigate the relative displacement of the trajectories $\varphi_\tau(p)$, $\varphi_\tau(q)$ of two (eventually infinitesimally close) nearby points p, $q \in \gamma$ in the "surface" $\{\tau = 0\}$:



$$\begin{aligned} p, q \in \{\tau = 0\} &\mapsto \varphi_\tau(p),\ \varphi_\tau(q)\ , \\ \gamma \subset \{\tau = 0\} &\mapsto \varphi_\tau \circ \gamma\ . \end{aligned}$$

Vector fields $n = d\gamma/ds$ ("infinitesimal displacements") in the surface $\{\tau = 0\}$ are mapped according to

$$n_p \mapsto \varphi_{\tau *} n_p =: n_{\varphi_\tau(p)}$$

(Lie transport) and thus extended to vector fields $n = \varphi_{\tau *} n$ on M. In particular we have

$$[u, n] = \frac{d}{d\tau} \varphi_{\tau *} n \Big|_{\tau=0} = 0 \ .$$

This implies $\nabla_u n = \nabla_n u$ (torsion $= 0$) and

$$\nabla_u^2 n = \nabla_u \nabla_n u = [R(u,n) + \nabla_n \nabla_u]\, u \ ,$$

i.e. we have the equation of **geodesic deviation**

$$\nabla_u^2 n = R(u,n) u \ . \tag{4.18}$$

The curvature describes the relative acceleration of nearby freely falling particles.

Remarks: 1) The choice of the surface $\{\tau = 0\}$ is irrelevant, since an infinitesimal change amounts to the replacement $n \rightsquigarrow u + \lambda n$ with $u\lambda = 0$; then we have $\nabla_u(\lambda u) = 0$ and $R(u, \lambda u) = 0$.

2) If the surface $\{\tau = 0\}$ is perpendicular to u, then we have

$$g(u,n) = 0$$

there, and hence everywhere, since by $\nabla g = 0$ one has

$$u[g\,(u,n)] = g(\underbrace{\nabla_u u}_{=0}, n) + g(u, \underbrace{\nabla_u n}_{=\nabla_n u}) = \frac{1}{2} n[\underbrace{g(u,u)}_{=c^2}] = 0 \ .$$

3) Let e_μ be a basis of vector fields with $[e_\mu, u] = 0$ and $e_0 = 0$. The relative acceleration in direction i, $(i = 1,2,3)$ of particles, whose separation is in the same direction, is $\langle e^i, \nabla_u^2 e_i \rangle$. Summed over directions we obtain

$$\sum_{i=1}^{3} \langle e^i, \nabla_u^2 e_i \rangle = \langle e^\mu, \nabla_u^2 e_\mu \rangle = \langle e^\mu, R(u, e_\mu) u \rangle = -\mathrm{Ric}(u,u) \ . \tag{4.19}$$

4) The geodesic deviation in Newtonian mechanics is found by differentiating $\ddot{x}^i = -\varphi_{,i}(x)$ w.r.t. s, where $n^i = \partial x^i / \partial s|_{s=0}$. This yields

$$\ddot{n}^i = -\varphi_{,ik} n^k \ . \tag{4.20}$$