Exercise Sheet IX

Problem 1 [*Self-mappings of the sphere* $\hat{\mathbb{C}}$.]:

(a) A fixed point of a transformation $w = f(z)$ is a point z_0 such that $f(z_0) = z_0$. Prove that each linear fractional transformation, with the exception of the identity transformation $w = z$, has at most two fixed points.

(b) The equation

$$\frac{z - z_1}{z - z_2}\frac{z_3 - z_2}{z_3 - z_1} = \frac{w - w_1}{w - w_2}\frac{w_3 - w_2}{w_3 - w_1}.$$

implicitly defines a map $w(z)$. Show that it is the unique linear fractional transformation that maps the three different points z_1, z_2 and z_3 into three different points w_1, w_2 and w_3, respectively. [Hint: Let S and T be two distinct linear fractional transformations which satisfy this property, and consider the transformation obtained by composing S with the inverse of T.]

Problem 2 [*Self-mappings of* $\bar{\mathbb{H}}$.]: Let $w(z)$ be the linear fractional transformation

$$w = \frac{az + b}{cz + d}.$$

(a) Prove that if $w(z)$ maps the real line of the z plane into the real line of the w plane, then a, b, c and d must all be real, except possibly for a common phase factor that can be removed without changing the transformation.

(b) Show that (a) implies that $w(z)$ maps the upper half z plane to the upper half w plane if and only if $ad - bc > 1$.

Problem 3 [*Upper half-plane* $\bar{\mathbb{H}}$ *with two boundary punctures.*]: Consider $\bar{\mathbb{H}}$ with two punctures P_1 and P_2 on the real line, with coordinates $z = x_1$ and $z = x_2$, respectively. Consider another copy of $\bar{\mathbb{H}}$ with two punctures P_1 and P_2 on the real line, with coordinates $z = x_1'$ and $z = x_2'$, respectively. Are these two surfaces the same Riemann surface? Prove that they are, by exhibiting the conformal map that takes the punctures into each other while preserving $\bar{\mathbb{H}}$. You may have to write two conformal maps, depending on the sign of $(x_2' - x_1')/(x_2 - x_1)$. What is the geometrical significance of this sign?

Problem 4 [*Closing off the polygon in the Schwarz-Christoffel map.*]: The differential equation

$$\frac{dw}{dz} = A(z - x_1)^{-\frac{\alpha_1}{\pi}}(z - x_2)^{-\frac{\alpha_2}{\pi}} \cdots (z - x_{n-1})^{-\frac{\alpha_{n-1}}{\pi}} \tag{1}$$

does not show the turning angle α_n at P_n because this point has been mapped to $z = \infty$. We aim to understand how this point at infinity works out.

(a) To find the turning angle at $z = \infty$, consider the large z-limit of (1):

$$\frac{dw}{dz} \simeq Az^{-\frac{1}{\pi}\sum_{i=1}^{n-1}\alpha_i}.$$

Define $t = 1/z$, and calculate $\frac{dw}{dt}$ as a function of t. Explain why your result shows that the turning angle is α_n.

(b) The differential equation

$$\frac{dw}{dz} = A(z - x_1)^{-\frac{\alpha_1}{\pi}}(z - x_2)^{-\frac{\alpha_2}{\pi}} \cdots (z - x_n)^{-\frac{\alpha_n}{\pi}}, \tag{2}$$

with the last turning point P_n included as a finite point x_n, represents the situation where there is no corner at $z = \infty$. Prove that the polygon closes. For this, show that as we traverse the full real axis x, the change in w is zero:

$$w(x = \infty) - w(x = -\infty) = \int_{x=-\infty}^{x=\infty} dx \frac{dw}{dx} = 0.$$

[Hint: Use (2) and contour deformation. Argue that there is no contribution from half-circles around the x_i and around ∞.]