Exercise Sheet VIII

Hand in by 10.12.2008

Problem 1 [*Zero mode Hamiltonian.*]: We can use the action

$$S = \int d\tau d\sigma \mathcal{L} = \frac{1}{4\pi\alpha'} \int d\tau \int_0^\pi d\sigma \left(\dot{X}\dot{X} - X'X' \right)$$

to study the dynamics of the compact coordinate X. Consider the zero mode expansion of the string coordinate in the sector of winding number m:

$$X(\tau, \sigma) = x(\tau) + mR\sigma.$$

Find the action for $x(\tau)$. Calculate the Hamiltonian, and show that you recover the contributions to

$$H = \frac{\alpha'}{2}(p^i p^i + p^2 + w^2) + N^\perp + \bar{N}^\perp - 2$$

arising from the compact dimension.

Problem 2 [*Compactification on T^2 with a constant Kalb-Ramond field.*]: Assume that x^2 and x^3 are each compactified into a circle of radius R. The corresponding string coordinates are called X^r, with $r = 2, 3$. Moreover, there is a non-vanishing Kalb-Ramond field with expectation value

$$B_{23} \equiv \frac{1}{2\pi\alpha'} b,$$

where b is a dimensionless constant. All other components of $B_{\mu\nu}$ vanish.

(a) Build an action for the $X^r(\tau, \sigma)$ by adding an action of the type used in Problem 1 to the action

$$S = -\int d\tau d\sigma \frac{\partial X^\mu}{\partial \tau} \frac{\partial X^\nu}{\partial \sigma} B_{\mu\nu}(X).$$

(b) Consider the following expansion for the zero mode part of the coordinates:

$$X^r = x^r(\tau) + m_r R\sigma, \quad r = 2, 3.$$

Show that the Lagrangian for $x^r(\tau)$ is

$$L = \frac{1}{2\alpha'} \left((\dot{x}^2)^2 + (\dot{x}^3)^2 \right) - \frac{\alpha'}{2} \left((w_2)^2 + (w_3)^2 \right) - b \left(\dot{x}^2 w_3 - \dot{x}^3 w_2 \right). \tag{1}$$

Here $w_r = m_r R/\alpha'$. The last term in (1) is a total derivative, but it is important in the quantum theory, as you will see.

(c) Define momenta canonical to x^r, compute the Hamiltonian, and show that it takes the form

$$H = \frac{\alpha'}{2}\left((p_2 + bw_3)^2 + (p_3 - bw_2)^2 + w_2^2 + w_3^2\right).$$

Verify that the Hamiltonian generates the correct equations of motion. Note that the quantisation conditions on the momenta are $p_r = n_r/R$.

(d) While we have only looked explicitly at the zero modes, the oscillator expansion of the coordinates works just as before. Write the appropriate expansions for the coordinates $X^2(\tau,\sigma)$ and $X^3(\tau,\sigma)$ along the lines of

$$X(\tau,\sigma) = x_0 + \alpha' p\tau + \alpha' w\sigma + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{e^{-in\tau}}{n}(\bar{\alpha}_n e^{-in\sigma} + \alpha_n e^{in\sigma}).$$

Explain why the mass-squared operator takes the form

$$\begin{aligned} M^2 = {} & \left(\frac{n_2}{R} + b\frac{m_3 R}{\alpha'}\right)^2 + \left(\frac{n_3}{R} - b\frac{m_2 R}{\alpha'}\right)^2 \\ & + \left(\frac{m_2 R}{\alpha'}\right)^2 + \left(\frac{m_3 R}{\alpha'}\right)^2 + \frac{2}{\alpha'}\left(N^\perp + \bar{N}^\perp - 2\right). \end{aligned} \tag{2}$$

(e) Show that the constraint $L_0^\perp - \bar{L}_0^\perp = 0$ yields

$$N^\perp - \bar{N}^\perp = n_2 m_2 + n_3 m_3. \tag{3}$$

Problem 3 [*Dualities in the T^2 compactification with Kalb-Ramond field.*]: In problem 2 you obtained equations (2) and (3), which define the spectrum of a compactification on a square torus T^2 with radius R and with Kalb-Ramond field $B_{23} = b/(2\pi\alpha')$. We are now interested in duality symmetries. Show that the spectrum is unchanged under the changes in the background parameters R and b listed below.

(a) The value of b is changed as

$$b \mapsto b' = b + \ell\frac{\alpha'}{R^2}, \quad \ell \in \mathbb{Z}, \tag{4}$$

while R is left unchanged. Equation (4) states that b is a *periodic* variable. Let A denote the area of the torus. Use (4) to show that the "flux" parameter $f_B = B_{23}A$ is an *angle* variable (i.e., $f_b \sim f_b + 2\pi$). To prove that the spectrum is unchanged, you must find an appropriate compensating change in the quantum numbers, as in

$$M^2(R; n, m) = M^2(\tilde{R}; m, n),$$

for example. [Hint: $n_2 \to n_2 - \ell m_3$ is one needed change.]

(b) The values of R and b are changed as

$$R \mapsto R' = \frac{\alpha'}{R} \frac{1}{\sqrt{1+b^2}}, \qquad b \mapsto b' = -b.$$

When $b = 0$ this is the familiar T-duality transformation of the radius. The compensating change here is the expected $n_r \leftrightarrow m_r$. [Hint: The algebra is easier if you first expand (2).]

(c) $b \mapsto -b$, with R unchanged. Use $m_r \mapsto -m_r$ as the compensating change of quantum numbers. What additional change must be done regarding oscillators to make (3) work?