Exercise Sheet I

Hand in by 01.10.2008

Problem 1 [*Mixed boundary conditions.*]: We consider the transverse oscillations of a non-relativistic string stretched from $x = 0$ to $x = a$ with tension T_0 and constant mass density μ_o. The transverse oscillations are in the y-direction. Make the ansatz

$$y(t, x) = \psi(x)\,\sin(\omega t + \phi)$$

to solve the differential equation

$$\frac{\partial^2 y}{\partial x^2} - \frac{\mu_0}{T_0}\frac{\partial^2 y}{\partial t^2} = 0.$$

Determine the possible frequencies for the Dirichlet condition $y = 0$ at $x = 0$ and the Neumann condition $\frac{\partial y}{\partial x} = 0$ at $x = a$.

Problem 2 [*A configuration with two joined strings.*]: A string with tension T_0 is stretched from $x = 0$ to $x = 2a$. The part of the string $x \in (0, a)$ has constant mass density μ_1, and the part of the string $x \in (a, 2a)$ has constant mass density μ_2. Consider the differential equation

$$\frac{d^2\psi}{dx^2} + \frac{\mu(x)}{T_0}\omega^2\psi(x) = 0$$

that determines the normal oscillations.

(a) What boundary conditions should be imposed on $\psi(x)$ and $\frac{d\psi}{dx}(x)$ at $x = a$?

(b) Write the conditions that determine the possible frequencies of oscillation.

(c) Calculate the lowest frequency of oscillation of this string when $\mu_2 = 2\mu_1$.

Problem 3 [*Variational problem for strings.*]: Consider a string stretched from $x = 0$ to $x = a$, with tension T_0 and a position-dependent mass density $\mu(x)$. The string is fixed at the endpoints and can vibrate in the y direction. The equation

$$\frac{d^2\psi}{dx^2} + \frac{\mu(x)}{T_0}\omega^2\psi(x) = 0$$

determines the oscillation frequencies ω_i and associated profiles $\psi_i(x)$ for the string.

(a) Set up a variational procedure that gives an upper bound on the lowest frequency of oscillation ω_0. (This can be done roughly as in quantum mechanics, where the ground state energy E_0 of a system with Hamiltonian H satisfies $E_0 \leq \langle\psi, H\psi\rangle / \langle\psi, \psi\rangle$.) As a useful first step consider the inner product

$$\langle\psi_i, \psi_j\rangle := \int_0^a \mu(x)\psi_i(x)\psi_j(x)\,dx$$

and show that is vanishes when $\omega_i \neq \omega_j$. Explain why your variational procedure works.

(b) Consider the case $\mu(x) = \mu_0 \frac{x}{a}$. Use your variational principle to find a simple bound on the lowest oscillation frequency. Compare with the answer $\omega_0^2 \simeq 18.956 \frac{T_0}{\mu_0 a^2}$ obtained by a direct numerical solution of the eigenvalue problem.