Sheet 10

Due date: 27 May 2014

Exercise 1 [*Angular momentum*]: The angular momentum operator is defined by

$$\mathbf{L} = \mathbf{x} \wedge \mathbf{p} \ ,$$

and thus its components are

$$L_i = \varepsilon_{ijk}\, x_j\, p_k \ .$$

(i) Using the fact that $[x_i, p_j] = i\hbar\delta_{ij}$, derive the commutation relations for the L_i

$$[L_i, L_j] = i\hbar\varepsilon_{ijk} L_k \ . \tag{1}$$

(ii) Use (1) to show that

$$[\mathbf{L}^2, L_j] = 0 \qquad \text{for } j = 1, 2, 3 \ .$$

Let us denote by $|l, m\rangle$ the eigenstates of both $\mathbf{L}^2$ and L_3 such that

$$\begin{aligned} \mathbf{L}^2\ |l, m\rangle &= \hbar^2 l(l+1)\,|l, m\rangle \\ L_3 |l, m\rangle &= \hbar m\,|l, m\rangle \ . \end{aligned}$$

(iii) Evaluate the commutator $[L_3, L_1L_2 + L_2L_1]$, and deduce that the expectation values of L_1^2 and L_2^2 with respect to $|l, m\rangle$ are given by

$$\langle l, m|L_1^2|l, m\rangle = \langle l, m|L_2^2|l, m\rangle = \frac{1}{2}\hbar^2[l(l+1) - m^2] \ .$$

Hint: If ψ is an eigenstate of the self-adjoint operator $\mathbf{A}$, show that, for any operator $\mathbf{B}$,

$$\langle\psi|[\mathbf{A}, \mathbf{B}]\psi\rangle = 0.$$

Exercise 2 [*Oscillator representation of su(2)*]: Let $a^\dagger_\pm$ and $a_\pm$ be two pairs of creation and annihilation operators, i. e.

$$[a_+, a^\dagger_+] = [a_-, a^\dagger_-] = 1 \ ,$$

while all the other commutators vanish. Define

$$J_3 = \frac{1}{2}\big(a^\dagger_+ a_+ - a^\dagger_- a_-\big) \ , \qquad J_+ = a^\dagger_+ a_- \ , \qquad J_- = a^\dagger_- a_+ \ .$$

(i) Show that these operators satisfy the commutation relations of su(2) ,

$$[J_3, J_\pm] = \pm J_\pm \ , \qquad [J_+, J_-] = 2J_3 \ .$$

(ii) Calculate $\mathbf{J}^2 = J_3^2 + \frac{1}{2}(J_+J_- + J_-J_+)$ and show that it equals $\frac{N}{2}(\frac{N}{2}+1)$, where $N = a_+^\dagger a_+ + a_-^\dagger a_-$ is the (total) number operator.

(iii) Let us denote by $|n_+, n_-\rangle$ the eigenstates of the number operators $N_\pm = a_\pm^\dagger a_\pm$ with eigenvalues $n_\pm$. Show that

$$
\begin{aligned}
J_+|n_+,n_-\rangle &= \sqrt{n_-(n_+ + 1)}\,|n_+ + 1, n_- - 1\rangle \ , \\
J_-|n_+,n_-\rangle &= \sqrt{n_+(n_- + 1)}\,|n_+ - 1, n_- + 1\rangle \ , \\
J_3|n_+,n_-\rangle &= \frac{1}{2}(n_+ - n_-)\,|n_+,n_-\rangle \ .
\end{aligned}
$$

(iv) Recall that on the standard basis $|j,m\rangle$ the generators of su(2) act as

$$
\begin{aligned}
J_3|j,m\rangle &= m\,|j,m\rangle \ , \\
J_\pm\,|j,m\rangle &= \sqrt{j(j+1) - m(m\pm 1)}\,|j, m\pm 1\rangle \ .
\end{aligned}
$$

Show that the above operators take this form with

$$
j = \frac{1}{2}(n_+ + n_-) \ , \qquad \text{and} \qquad m = \frac{1}{2}(n_+ - n_-) \ .
$$

Thus conclude that we have the identification

$$
|j,m\rangle = \frac{(a_+^\dagger)^{j+m}\,(a_-^\dagger)^{j-m}}{\sqrt{(j+m)!(j-m)!}}|0,0\rangle \ .
$$