Concept of Landau's Fermi liquid theory

elementary excitations of interacting Fermions are described by almost independent fermionic quasiparticles

state of Fermi liquid described simply by quasiparticle distribution

Phenomenological Theory by Landau

energy functional:

$$E = E_0 + \sum_{\vec{k},\sigma} \epsilon_\sigma(\vec{k})\delta n_\sigma(\vec{k}) + \frac{1}{2\Omega}\sum_{\vec{k},\vec{k}'}\sum_{\sigma,\sigma'} f_{\sigma\sigma'}(\vec{k},\vec{k}')\delta n_\sigma(\vec{k})\delta n_{\sigma'}(\vec{k}')$$

deviation from ground state

$$\delta n_\sigma(\vec{k}) = n_\sigma(\vec{k}) - n_\sigma^{(0)}(\vec{k})$$

spin index $\sigma = \pm 1$

ground state distribution

$$n_\sigma^{(0)}(\vec{k}) = \Theta(k_F - |\vec{k}|)$$

filled Fermi sea

Fermi liquid theory

$$E = E_0 + \sum_{\vec{k},\sigma} \epsilon_\sigma(\vec{k})\delta n_\sigma(\vec{k}) + \frac{1}{2\Omega} \sum_{\vec{k},\vec{k}'} \sum_{\sigma,\sigma'} f_{\sigma\sigma'}(\vec{k},\vec{k}')\delta n_\sigma(\vec{k})\delta n_{\sigma'}(\vec{k}')$$

effective quasiparticle spectrum:

$$\tilde{\epsilon}_\sigma(\vec{k}) = \frac{\delta E}{\delta n_\sigma(\vec{k})} = \epsilon_\sigma(\vec{k}) + \frac{1}{\Omega} \sum_{\vec{k}',\sigma'} f_{\sigma\sigma'}(\vec{k},\vec{k}')\delta n_{\sigma'}(\vec{k}')$$

bare quasiparticle spectrum:

$$\epsilon_\sigma(\vec{k}) = \frac{\hbar^2 \vec{k}^2}{2m^*}$$

effective mass

Fermi velocity:

$$\frac{1}{\hbar}\vec{\nabla}_{\vec{k}}\epsilon_\sigma(\vec{k})\Big|_{k_F} = \vec{v}_F = \frac{\hbar \vec{k}_F}{m^*}$$

density of states at ϵ_F :

$$N(\epsilon_F) = \frac{1}{\Omega} \sum_{\vec{k},\sigma} \delta(\epsilon_\sigma(\vec{k}) - \epsilon_F) = \frac{k_F^2}{\pi^2 \hbar v_F} = \frac{m^* k_F}{\pi^2 \hbar^2}$$

Fermi volume conserved

$$k_F = (3\pi^2 n)^{1/3}$$

Fermi liquid theory

$$E = E_0 + \sum_{\vec{k},\sigma} \epsilon_\sigma(\vec{k})\delta n_\sigma(\vec{k}) + \frac{1}{2\Omega}\sum_{\vec{k},\vec{k}'}\sum_{\sigma,\sigma'} f_{\sigma\sigma'}(\vec{k},\vec{k}')\delta n_\sigma(\vec{k})\delta n_{\sigma'}(\vec{k}')$$

couplings:

$$f_{\sigma\sigma'}(\vec{k},\vec{k}') = f^s(\hat{k},\hat{k}') + \sigma\sigma' f^a(\hat{k},\hat{k}')$$

symmetric (charge) — antisymmetric (spin)

spherical symmetry: $f^{s,a}(\hat{k},\hat{k}') = \sum_{l=0}^{\infty} f_l^{s,a} P_l(\cos\theta_{\hat{k},\hat{k}'})$ Legendre Polynomials

$$\int_{-1}^{+1} dz\, P_l(z)\, P_{l'}(z) = \frac{2\delta_{ll'}}{2l+1}$$

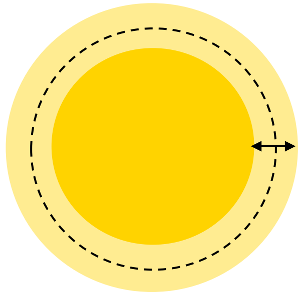
Landau parameters:

$$F_l^s = N(\epsilon_F) f_l^s \quad \text{charge}$$

$$F_l^a = N(\epsilon_F) f_l^a \quad \text{spin}$$

Fermi liquid theory - physical properties

specific heat: $$\delta n_\sigma(\vec{k}) = n_\sigma^{(0)}(T, \vec{k}) - n_\sigma^{(0)}(0, \vec{k})$$

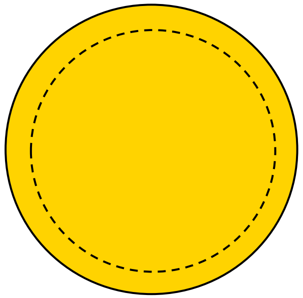


T

thermal softening
of Fermi surface

$$\rightarrow \quad C = \frac{\pi^2 k_B^2 N(\epsilon_F)}{3} T$$

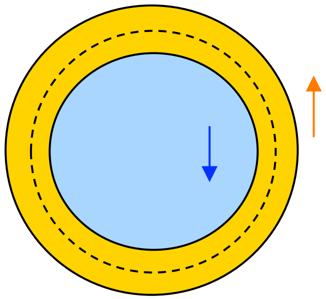
compressibility: $$\kappa = -\frac{1}{\Omega} \frac{\partial \Omega}{\partial p}\bigg|_{T,N} \qquad \delta\tilde{\epsilon}_\sigma(\vec{k}) = \frac{2}{3}\tilde{\epsilon}_\sigma(\vec{k})\kappa\delta p$$



change of
Fermi volume

$$\rightarrow \quad \kappa = \frac{3}{2n\epsilon_F}\frac{1}{1+F_0^s} = \frac{1}{n^2}\frac{N(\epsilon_F)}{1+F_0^s}$$

spin susceptibility: $$\chi = \frac{M}{H} \qquad \delta\tilde{\epsilon}_\sigma(\vec{k}) = -\tilde{g}\mu_B H \frac{\sigma}{2}$$

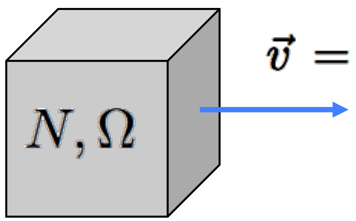


spin splitting
of Fermi sea

$$\rightarrow \quad \chi = \frac{\mu_B^2 N(\epsilon_F)}{1+F_0^a}$$

Fermi liquid theory - Gallilei invariance

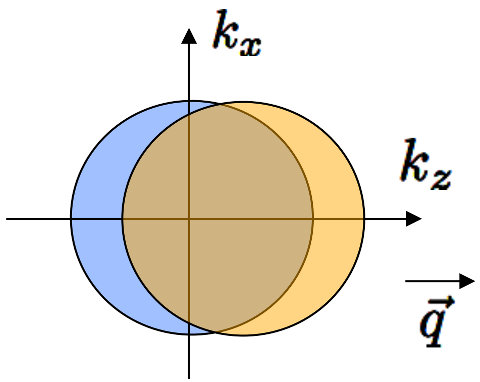
momentum shift for all particles $\vec{k} \rightarrow \vec{k} + \vec{q}$



$$\vec{v} = \frac{\hbar \vec{q}}{m}$$

current density

$$\vec{j}_{\vec{q}} = \frac{1}{\Omega} \sum_{\vec{k},\sigma} \vec{v}(\vec{k})\, \delta n_\sigma(\vec{k})$$



shifted Fermi sea

"bare particle" view

$$\vec{v}(\vec{k}) = \frac{\hbar \vec{k}}{m}$$

quasiparticle view

$$\vec{v}(\vec{k}) = \frac{1}{\hbar} \vec{\nabla}_{\vec{k}}\, \tilde{\epsilon}_\sigma(\vec{k})$$

$$= \frac{1}{\hbar}\left(\vec{\nabla}_{\vec{k}}\, \epsilon_\sigma(\vec{k}) + \frac{1}{\Omega} \sum_{\vec{k}',\sigma'} \vec{\nabla}_{\vec{k}}\, f_{\sigma\sigma'}(\vec{k}, \vec{k}')\delta n_\sigma(\vec{k}') \right)$$

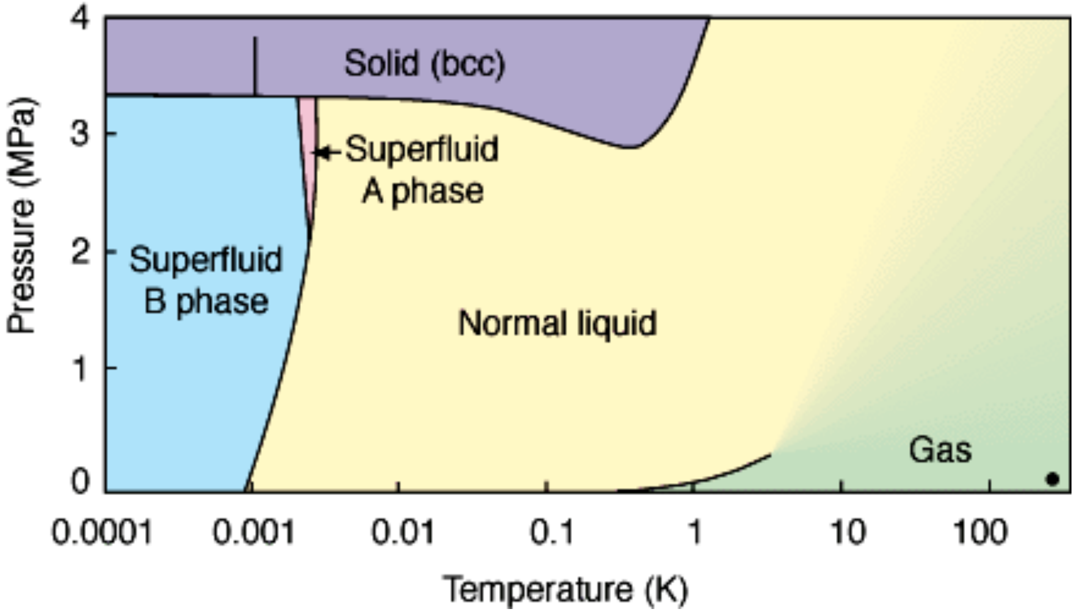
quasiparticle motion

induced motion

$$\frac{\hbar \vec{k}}{m} = \frac{\hbar \vec{k}}{m^*} + \frac{1}{\Omega} \sum_{\vec{k}',\sigma'} f_{\sigma\sigma'}(\vec{k}, \vec{k}')\delta(\epsilon_\sigma(\vec{k}') - \epsilon_F)\frac{\hbar \vec{k}'}{m^*}$$

$$\frac{m^*}{m} = 1 + \frac{1}{3} F_1^s$$

intrinsic consistency



strong short-range repulsion

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Fermi liquid

longer-ranged attraction

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superfluid

pressure	m^*/m	F_0^s	F_0^a	F_1^s	κ/κ_0	χ/χ_0
0	3.0	10.1	-0.52	6.0	0.27	6.3
$< p_c$	6.2	94	-0.74	15.7	0.065	24

m^*/m: enhanced; κ/κ_0: diminished; χ/χ_0: enhanced

Hamiltonian

$$\mathcal{H} = \sum_{\vec{k},s} \epsilon_{\vec{k}} \widehat{c}^{\dagger}_{\vec{k}s} \widehat{c}_{\vec{k}s} + \int d^3r\, d^3r'\, \widehat{\Psi}_{\uparrow}(\vec{r})^{\dagger} \widehat{\Psi}_{\downarrow}(\vec{r}')^{\dagger} U \delta(\vec{r}-\vec{r}') \widehat{\Psi}_{\downarrow}(\vec{r}') \widehat{\Psi}_{\uparrow}(\vec{r})$$

$$= \underbrace{\sum_{\vec{k},s} \epsilon_{\vec{k}} \widehat{c}^{\dagger}_{\vec{k}s} \widehat{c}_{\vec{k}s}}_{\text{kinetic energy}} + \underbrace{\frac{U}{\Omega} \sum_{\vec{k},\vec{k}',\vec{q}} \widehat{c}^{\dagger}_{\vec{k}+\vec{q}\uparrow} \widehat{c}^{\dagger}_{\vec{k}'-\vec{q}\downarrow} \widehat{c}_{\vec{k}'\downarrow} \widehat{c}_{\vec{k}\uparrow}}_{\text{contact interaction}}$$

kinetic energy

$$\epsilon_{\vec{k}} = \frac{\hbar^2 \vec{k}^2}{2m}$$

contact interaction
„perturbation“

Rayleigh-Schrödinger perturbation theory → **Fermi liquid theory**
Landau parameters

Rayleigh-Schrödinger

energy: $$E = E^{(0)} + E^{(1)} + E^{(2)} + \cdots$$

$$E^{(0)} = \sum_{\vec{k},s} \epsilon_{\vec{k}} n_{\vec{k}s} \qquad E^{(1)} = \frac{U}{\Omega} \sum_{\vec{k},\vec{k}'} n_{\vec{k}\uparrow} n_{\vec{k}'\downarrow}$$

$$E^{(2)} = \frac{U^2}{\Omega^2} \sum_{\vec{k},\vec{k}',\vec{q}} \frac{n_{\vec{k}\uparrow} n_{\vec{k}'\downarrow}(1 - n_{\vec{k}+\vec{q}\uparrow})(1 - n_{\vec{k}'-\vec{q}\downarrow})}{\epsilon_{\vec{k}} + \epsilon_{\vec{k}'} - \epsilon_{\vec{k}+\vec{q}} - \epsilon_{\vec{k}'-\vec{q}}}$$

Distribution function
characterizes unperturbed state

$$n_{\vec{k}s} = \langle \hat{c}^{\dagger}_{\vec{k}s} \hat{c}_{\vec{k}s} \rangle = n^{(0)}_{\vec{k}s} + \delta n_{\vec{k}s}$$

Rayleigh-Schrödinger

$$n_{\vec{k}s} = \langle \hat{c}^\dagger_{\vec{k}s} \hat{c}_{\vec{k}s} \rangle = n^{(0)}_{\vec{k}s} + \delta n_{\vec{k}s}$$

$$E^{(1)} = \frac{U}{\Omega} \sum_{\vec{k},\vec{k}'} n_{\vec{k}\uparrow} n_{\vec{k}'\downarrow}$$

$$E^{(2)} = \frac{U^2}{\Omega^2} \sum_{\vec{k},\vec{k}',\vec{q}} \frac{n_{\vec{k}\uparrow} n_{\vec{k}'\downarrow} - n_{\vec{k}\uparrow} n_{\vec{k}'\downarrow}(n_{\vec{k}+\vec{q}\uparrow} + n_{\vec{k}'-\vec{q}\downarrow}) + n_{\vec{k}\uparrow} n_{\vec{k}'\downarrow} n_{\vec{k}+\vec{q}\uparrow} n_{\vec{k}'-\vec{q}\downarrow}}{\epsilon_{\vec{k}} + \epsilon_{\vec{k}'} - \epsilon_{\vec{k}+\vec{q}} - \epsilon_{\vec{k}'-\vec{q}}}$$

$$\tilde{E}^{(1)} = E^{(1)} + \frac{U^2}{\Omega^2} \sum_{\vec{k},\vec{k}',\vec{q}} \frac{n_{\vec{k}\uparrow} n_{\vec{k}'\downarrow}}{\epsilon_{\vec{k}} + \epsilon_{\vec{k}'} - \epsilon_{\vec{k}+\vec{q}} - \epsilon_{\vec{k}'-\vec{q}}} \approx \frac{\tilde{U}}{\Omega} \sum_{\vec{k},\vec{k}'} n_{\vec{k}\uparrow} n_{\vec{k}'\downarrow}$$

$$\tilde{U} = U + \frac{U^2}{\Omega} \sum_{\vec{q}} \frac{1}{\epsilon_{\vec{k}} + \epsilon_{\vec{k}'} - \epsilon_{\vec{k}+\vec{q}} - \epsilon_{\vec{k}'-\vec{q}}}$$

Rayleigh-Schrödinger

$$n_{\vec{k}\,s} = \langle \hat{c}^{\dagger}_{\vec{k}\,s} \hat{c}_{\vec{k}\,s} \rangle = n^{(0)}_{\vec{k}\,s} + \delta n_{\vec{k}\,s}$$

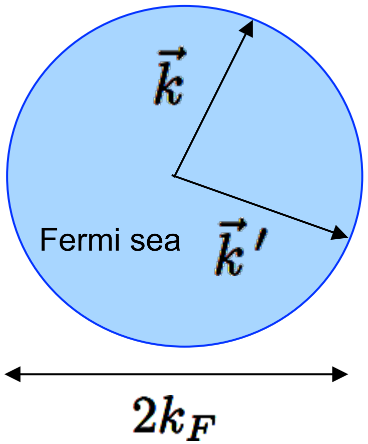
$$\tilde{U} = U + \frac{U^2}{\Omega} \sum_{\vec{q}} \frac{1}{\epsilon_{\vec{k}} + \epsilon_{\vec{k}\,'} - \epsilon_{\vec{k}+\vec{q}} - \epsilon_{\vec{k}\,'-\vec{q}}}$$

$$\frac{1}{\Omega} \sum_{\vec{q}} \frac{1}{\epsilon_{\vec{k}} + \epsilon_{\vec{k}\,'} - \epsilon_{\vec{k}+\vec{q}} - \epsilon_{\vec{k}\,'-\vec{q}}} = \frac{1}{(2\pi)^3} \int_o^{Q_c} dq\, q^2 \int d\Omega_q \frac{m}{(\vec{k}\,' - \vec{k}) \cdot \vec{q} - \vec{q}^{\,2}}$$

$$= \frac{m}{(2\pi)^2} \int dq\, q \int_{-1}^{+1} \frac{d\cos\theta}{K\cos\theta - q}$$

$$\vec{K} = \vec{k}\,' - \vec{k}$$

$$K = |\vec{k}\,' - \vec{k}| \leq 2k_F$$



Rayleigh-Schrödinger

$$n_{\vec{k}\,s} = \langle \hat{c}^{\dagger}_{\vec{k}\,s} \hat{c}_{\vec{k}\,s} \rangle = n^{(0)}_{\vec{k}\,s} + \delta n_{\vec{k}\,s}$$

$$\tilde{U} = U + \frac{U^2}{\Omega} \sum_{\vec{q}} \frac{1}{\epsilon_{\vec{k}} + \epsilon_{\vec{k}'} - \epsilon_{\vec{k}+\vec{q}} - \epsilon_{\vec{k}'-\vec{q}}} \quad < U \quad \text{„screening“}$$

$$\frac{1}{\Omega} \sum_{\vec{q}} \frac{1}{\epsilon_{\vec{k}} + \epsilon_{\vec{k}'} - \epsilon_{\vec{k}+\vec{q}} - \epsilon_{\vec{k}'-\vec{q}}} = \frac{1}{(2\pi)^3} \int_o^{Q_c} dq\, q^2 \int d\Omega_q \frac{m}{(\vec{k}' - \vec{k}) \cdot \vec{q} - \vec{q}^{\,2}}$$

$$= \frac{m}{(2\pi)^2} \int dq\, q \int_{-1}^{+1} \frac{d\cos\theta}{K\cos\theta - q}$$

$$= \frac{m}{(2\pi)^2} \int_0^{Q_c} dq\, q \ln \left| \frac{q-K}{q+K} \right|$$

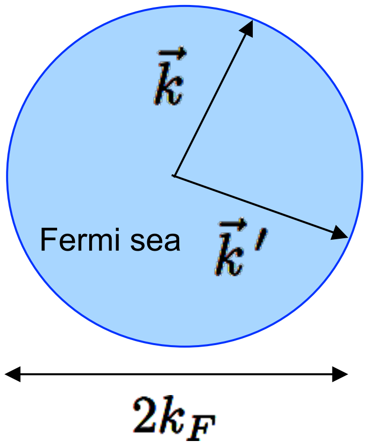
$$= -\frac{m}{(2\pi)^2} \left\{ Q_c + \frac{K^2 - Q_c^2}{2K} \ln \left| \frac{Q_c - K}{Q_c + K} \right| \right\}$$

$$\approx -\frac{2mQ_c}{(2\pi)^2} \left\{ 1 - \frac{K^2}{Q_c^2} + O\left(\frac{K^4}{Q_c^4} \right) \right\} \qquad K \ll Q_c$$

weak K-dependence

$$\vec{K} = \vec{k}' - \vec{k}$$

$$K = |\vec{k}' - \vec{k}| \leq 2k_F$$



cutoff

$Q_c \sim \ell^{-1}$

interaction range

ℓ

Rayleigh-Schrödinger $n_{\vec{k}s} = \langle \hat{c}^\dagger_{\vec{k}s} \hat{c}_{\vec{k}s} \rangle = n^{(0)}_{\vec{k}s} + \delta n_{\vec{k}s}$

$$E^{(2)} = \frac{U^2}{\Omega^2} \sum_{\vec{k},\vec{k}',\vec{q}} \frac{n_{\vec{k}\uparrow} n_{\vec{k}'\downarrow} - n_{\vec{k}\uparrow} n_{\vec{k}'\downarrow}(n_{\vec{k}+\vec{q}\uparrow} + n_{\vec{k}'-\vec{q}\downarrow}) + n_{\vec{k}\uparrow} n_{\vec{k}'\downarrow} n_{\vec{k}+\vec{q}\uparrow} n_{\vec{k}'-\vec{q}\downarrow}}{\epsilon_{\vec{k}} + \epsilon_{\vec{k}'} - \epsilon_{\vec{k}+\vec{q}} - \epsilon_{\vec{k}'-\vec{q}}}$$

cancels

relabeling $\left\{ \begin{array}{c} \vec{k} \leftrightarrow \vec{k} + \vec{q} \\ \vec{k}' \leftrightarrow \vec{k}' - \vec{q} \end{array} \right\}$

Rayleigh-Schrödinger $n_{\vec{k}s} = \langle \hat{c}^{\dagger}_{\vec{k}s} \hat{c}_{\vec{k}s} \rangle = n^{(0)}_{\vec{k}s} + \delta n_{\vec{k}s}$

$$E^{(2)} = \frac{U^2}{\Omega^2} \sum_{\vec{k},\vec{k}',\vec{q}} \frac{n_{\vec{k}\uparrow} n_{\vec{k}'\downarrow} - n_{\vec{k}\uparrow} n_{\vec{k}'\downarrow} (n_{\vec{k}+\vec{q}\uparrow} + n_{\vec{k}'-\vec{q}\downarrow}) + n_{\vec{k}\uparrow} n_{\vec{k}'\downarrow} n_{\vec{k}+\vec{q}\uparrow} n_{\vec{k}'-\vec{q}\downarrow}}{\epsilon_{\vec{k}} + \epsilon_{\vec{k}'} - \epsilon_{\vec{k}+\vec{q}} - \epsilon_{\vec{k}'-\vec{q}}}$$

$$\tilde{E}^{(2)} = -\frac{\tilde{U}^2}{\Omega^2} \sum_{\vec{k},\vec{k}',\vec{q}} \frac{n_{\vec{k}\uparrow} n_{\vec{k}'\downarrow} (n_{\vec{k}+\vec{q}\uparrow} + n_{\vec{k}'-\vec{q}\downarrow})}{\epsilon_{\vec{k}} + \epsilon_{\vec{k}'} - \epsilon_{\vec{k}+\vec{q}} - \epsilon_{\vec{k}'-\vec{q}}}$$

$$\tilde{U} = U + \frac{U^2}{\Omega} \sum_{\vec{q}} \frac{1}{\epsilon_{\vec{k}} + \epsilon_{\vec{k}'} - \epsilon_{\vec{k}+\vec{q}} - \epsilon_{\vec{k}'-\vec{q}}}$$

Rayleigh-Schrödinger $E = E^{(0)} + \tilde{E}^{(1)} + \tilde{E}^{(2)} + \cdots$

$$E^{(0)} = \sum_{\vec{k},s} \epsilon_{\vec{k}} n_{\vec{k}s} \qquad \tilde{E}^{(1)} = \frac{\tilde{U}}{\Omega} \sum_{\vec{k},\vec{k}'} n_{\vec{k}\uparrow} n_{\vec{k}'\downarrow}$$

$$\tilde{E}^{(2)} = -\frac{\tilde{U}^2}{\Omega^2} \sum_{\vec{k},\vec{k}',\vec{q}} \frac{n_{\vec{k}\uparrow} n_{\vec{k}'\downarrow} (n_{\vec{k}+\vec{q}\uparrow} + n_{\vec{k}'-\vec{q}\downarrow})}{\epsilon_{\vec{k}} + \epsilon_{\vec{k}'} - \epsilon_{\vec{k}+\vec{q}} - \epsilon_{\vec{k}'-\vec{q}}}$$

variation: $\tilde{\epsilon}_{\uparrow}(\vec{k}) = \frac{\delta E}{\delta n_{\vec{k}\uparrow}}$ $\qquad n_{\vec{k}s} = \langle \hat{c}^{\dagger}_{\vec{k}s} \hat{c}_{\vec{k}s} \rangle = n^{(0)}_{\vec{k}s} + \delta n_{\vec{k}s}$

$$\tilde{\epsilon}_{\uparrow}(\vec{k}) = \epsilon_{\vec{k}} + \frac{\tilde{U}}{\Omega} \sum_{\vec{k}'} n_{\vec{k}'\downarrow} - \frac{\tilde{U}^2}{\Omega^2} \sum_{\vec{k}',\vec{q}} \frac{n_{\vec{k}'\downarrow}(n_{\vec{k}+\vec{q}\uparrow} + n_{\vec{k}'-\vec{q}\downarrow}) - n_{\vec{k}+\vec{q}\uparrow} n_{\vec{k}'-\vec{q}\downarrow}}{\epsilon_{\vec{k}} + \epsilon_{\vec{k}'} - \epsilon_{\vec{k}+\vec{q}} - \epsilon_{\vec{k}'-\vec{q}}}$$

$$\tilde{\epsilon}_\sigma(\vec{k}) = \epsilon_\sigma(\vec{k}) + \frac{1}{\Omega} \sum_{\vec{k}',\sigma'} f_{\sigma\sigma'}(\vec{k},\vec{k}')\delta n_{\vec{k}\sigma'}$$

$$n_{\vec{k}s} = n^{(0)}_{\vec{k}s} + \delta n_{\vec{k}s}$$

$$\tilde{\epsilon}_\uparrow(\vec{k}) = \epsilon_{\vec{k}} + \frac{\tilde{U}}{\Omega}\sum_{\vec{k}'} n_{\vec{k}'\downarrow} - \frac{\tilde{U}^2}{\Omega^2}\sum_{\vec{k}',\vec{q}} \frac{n_{\vec{k}'\downarrow}(n_{\vec{k}+\vec{q}\uparrow} + n_{\vec{k}'-\vec{q}\downarrow}) - n_{\vec{k}+\vec{q}\uparrow}n_{\vec{k}'-\vec{q}\downarrow}}{\epsilon_{\vec{k}} + \epsilon_{\vec{k}'} - \epsilon_{\vec{k}+\vec{q}} - \epsilon_{\vec{k}'-\vec{q}}}$$

$$\tilde{\epsilon}_\sigma(\vec{k}) = \epsilon_\sigma(\vec{k}) + \frac{1}{\Omega}\sum_{\vec{k}',\sigma'} f_{\sigma\sigma'}(\vec{k},\vec{k}')\delta n_{\vec{k}\sigma'}$$

$$n_{\vec{k}s} = n^{(0)}_{\vec{k}s} + \delta n_{\vec{k}s}$$

$$\tilde{\epsilon}_\uparrow(\vec{k}) = \epsilon_{\vec{k}} + \frac{\tilde{U}}{\Omega}\sum_{\vec{k}'} n_{\vec{k}'\downarrow} - \frac{\tilde{U}^2}{\Omega^2}\sum_{\vec{k}',\vec{q}} \frac{n_{\vec{k}'\downarrow}(n_{\vec{k}+\vec{q}\uparrow} + n_{\vec{k}'-\vec{q}\downarrow}) - n_{\vec{k}+\vec{q}\uparrow} n_{\vec{k}'-\vec{q}\downarrow}}{\epsilon_{\vec{k}} + \epsilon_{\vec{k}'} - \epsilon_{\vec{k}+\vec{q}} - \epsilon_{\vec{k}'-\vec{q}}}$$

$$f_{\uparrow\uparrow}(\vec{k}_F, \vec{k}'_F)$$

$$\frac{\tilde{U}^2}{\Omega^2}\sum_{\vec{k}',\vec{q}} n_{\vec{k}+\vec{q}\uparrow} \frac{n_{\vec{k}'-\vec{q}\downarrow} - n_{\vec{k}'\downarrow}}{\epsilon_{\vec{k}} + \epsilon_{\vec{k}'} - \epsilon_{\vec{k}+\vec{q}} - \epsilon_{\vec{k}'-\vec{q}}}$$

$$\xrightarrow{\vec{k}+\vec{q}\to\vec{k}'_F} \frac{1}{\Omega}\sum_{\vec{k}'_F} n_{\vec{k}'_F\uparrow} \frac{\tilde{U}^2}{\Omega}\sum_{\vec{k}'} \left.\frac{n^{(0)}_{\vec{k}'-\vec{q}\downarrow} - n^{(0)}_{\vec{k}'\downarrow}}{\epsilon_{\vec{k}'} - \epsilon_{\vec{k}'-\vec{q}}}\right|_{\vec{q}=\vec{k}'_F-\vec{k}_F}$$

$$n_{\vec{k}s} = n^{(0)}_{\vec{k}s} + \delta n_{\vec{k}s}$$

$$= -\frac{1}{\Omega}\sum_{\vec{k}'_F} n_{\vec{k}'_F\uparrow} \frac{\tilde{U}^2}{2}\chi_0(\vec{k}'_F - \vec{k}_F)$$

Lindhard function

$$\tilde{\epsilon}_\sigma(\vec{k}) = \epsilon_\sigma(\vec{k}) + \frac{1}{\Omega}\sum_{\vec{k}',\sigma'} f_{\sigma\sigma'}(\vec{k},\vec{k}')\delta n_{\vec{k}\sigma'}$$

$$n_{\vec{k}s} = n^{(0)}_{\vec{k}s} + \delta n_{\vec{k}s}$$

$$\tilde{\epsilon}_\uparrow(\vec{k}) = \epsilon_{\vec{k}} + \frac{\tilde{U}}{\Omega}\sum_{\vec{k}'} n_{\vec{k}'\downarrow} - \frac{\tilde{U}^2}{\Omega^2}\sum_{\vec{k}',\vec{q}} \frac{n_{\vec{k}'\downarrow}(n_{\vec{k}+\vec{q}\uparrow} + n_{\vec{k}'-\vec{q}\downarrow}) - n_{\vec{k}+\vec{q}\uparrow} n_{\vec{k}'-\vec{q}\downarrow}}{\epsilon_{\vec{k}} + \epsilon_{\vec{k}'} - \epsilon_{\vec{k}+\vec{q}} - \epsilon_{\vec{k}'-\vec{q}}}$$

$$f_{\uparrow\uparrow}(\vec{k}_F, \vec{k}'_F) \qquad \frac{\tilde{U}^2}{\Omega^2}\sum_{\vec{k}',\vec{q}} n_{\vec{k}+\vec{q}\uparrow}\frac{n_{\vec{k}'-\vec{q}\downarrow} - n_{\vec{k}'\downarrow}}{\epsilon_{\vec{k}} + \epsilon_{\vec{k}'} - \epsilon_{\vec{k}+\vec{q}} - \epsilon_{\vec{k}'-\vec{q}}}$$

$$f_{\uparrow\uparrow}(\vec{k}_F, \vec{k}'_F) = f_{\downarrow\downarrow}(\vec{k}_F, \vec{k}'_F) = \frac{\tilde{U}^2}{2}\chi_0(\vec{k}_F - \vec{k}'_F)$$

$$n_{\vec{k}s} = n^{(0)}_{\vec{k}s} + \delta n_{\vec{k}s}$$

$$= -\frac{1}{\Omega}\sum_{\vec{k}'_F} n_{\vec{k}'_F\uparrow}\frac{\tilde{U}^2}{2}\chi_0(\vec{k}'_F - \vec{k}_F)$$

Lindhard function

$$\tilde{\epsilon}_\sigma(\vec{k}) = \epsilon_\sigma(\vec{k}) + \frac{1}{\Omega}\sum_{\vec{k}',\sigma'} f_{\sigma\sigma'}(\vec{k},\vec{k}')\delta n_{\vec{k}\sigma'} \qquad n_{\vec{k}s} = n^{(0)}_{\vec{k}s} + \delta n_{\vec{k}s}$$

$$\tilde{\epsilon}_\uparrow(\vec{k}) = \epsilon_{\vec{k}} + \frac{\tilde{U}}{\Omega}\sum_{\vec{k}'} n_{\vec{k}'\downarrow} - \frac{\tilde{U}^2}{\Omega^2}\sum_{\vec{k}',\vec{q}} \frac{n_{\vec{k}'\downarrow}(n_{\vec{k}+\vec{q}\uparrow} + n_{\vec{k}'-\vec{q}\downarrow}) - n_{\vec{k}+\vec{q}\uparrow}n_{\vec{k}'-\vec{q}\downarrow}}{\epsilon_{\vec{k}} + \epsilon_{\vec{k}'} - \epsilon_{\vec{k}+\vec{q}} - \epsilon_{\vec{k}'-\vec{q}}}$$

$$f_{\uparrow\downarrow}(\vec{k}_F, \vec{k}'_F)$$

$$\frac{\tilde{U}}{\Omega}\sum_{\vec{k}'} n_{\vec{k}'\downarrow} \quad - \frac{\tilde{U}^2}{\Omega^2}\sum_{\vec{k}',\vec{q}} n_{\vec{k}'\downarrow}\frac{n_{\vec{k}+\vec{q}\uparrow}}{\epsilon_{\vec{k}} + \epsilon_{\vec{k}'} - \epsilon_{\vec{k}+\vec{q}} - \epsilon_{\vec{k}'-\vec{q}}}$$

$$-\frac{\tilde{U}^2}{\Omega^2}\sum_{\vec{k}',\vec{q}} n_{\vec{k}'-\vec{q}\downarrow}\frac{n_{\vec{k}'\downarrow} - n_{\vec{k}+\vec{q}\uparrow}}{\epsilon_{\vec{k}} + \epsilon_{\vec{k}'} - \epsilon_{\vec{k}+\vec{q}} - \epsilon_{\vec{k}'-\vec{q}}}$$

$$\tilde{\epsilon}_\sigma(\vec{k}) = \epsilon_\sigma(\vec{k}) + \frac{1}{\Omega}\sum_{\vec{k}',\sigma'} f_{\sigma\sigma'}(\vec{k},\vec{k}')\delta n_{\vec{k}\sigma'} \qquad n_{\vec{k}s} = n^{(0)}_{\vec{k}s} + \delta n_{\vec{k}s}$$

$$\tilde{\epsilon}_\uparrow(\vec{k}) = \epsilon_{\vec{k}} + \frac{\tilde{U}}{\Omega}\sum_{\vec{k}'} n_{\vec{k}'\downarrow} - \frac{\tilde{U}^2}{\Omega^2}\sum_{\vec{k}',\vec{q}} \frac{n_{\vec{k}'\downarrow}(n_{\vec{k}+\vec{q}\uparrow} + n_{\vec{k}'-\vec{q}\downarrow}) - n_{\vec{k}+\vec{q}\uparrow}n_{\vec{k}'-\vec{q}\downarrow}}{\epsilon_{\vec{k}} + \epsilon_{\vec{k}'} - \epsilon_{\vec{k}+\vec{q}} - \epsilon_{\vec{k}'-\vec{q}}}$$

$$f_{\uparrow\downarrow}(\vec{k}_F,\vec{k}'_F)$$

$$f_{\uparrow\downarrow}(\vec{k}_F,\vec{k}'_F) = f_{\downarrow\uparrow}(\vec{k}_F,\vec{k}'_F) = \tilde{U} - \frac{\tilde{U}^2}{2}\left\{2\tilde{\chi}_0(\vec{k}_F + \vec{k}'_F) - \chi_0(\vec{k}_F - \vec{k}'_F)\right\}$$

$$\chi_0(\vec{k}'_F - \vec{k}_F) = \frac{1}{\Omega}\sum_{\vec{k}'} \frac{n^{(0)}_{\vec{k}'-\vec{q}\downarrow} - n^{(0)}_{\vec{k}'\downarrow}}{\epsilon_{\vec{k}'} - \epsilon_{\vec{k}'-\vec{q}}} \qquad \tilde{\chi}_0(\vec{q}) = \frac{1}{\Omega}\sum_{\vec{k}'} \frac{n^{(0)}_{\vec{k}'+\vec{q}\uparrow} + n^{(0)}_{\vec{k}'\downarrow}}{2\epsilon_F - \epsilon_{\vec{k}'+\vec{q}} - \epsilon_{\vec{k}'}}$$

$$\tilde{\epsilon}_\sigma(\vec{k}) = \epsilon_\sigma(\vec{k}) + \frac{1}{\Omega}\sum_{\vec{k}',\sigma'} f_{\sigma\sigma'}(\vec{k},\vec{k}')\delta n_{\vec{k}\sigma'} \qquad n_{\vec{k}s} = n^{(0)}_{\vec{k}s} + \delta n_{\vec{k}s}$$

$$\tilde{\epsilon}_\uparrow(\vec{k}) = \epsilon_{\vec{k}} + \frac{\tilde{U}}{\Omega}\sum_{\vec{k}'} n_{\vec{k}'\downarrow} - \frac{\tilde{U}^2}{\Omega^2}\sum_{\vec{k}',\vec{q}} \frac{n_{\vec{k}'\downarrow}(n_{\vec{k}+\vec{q}\uparrow} + n_{\vec{k}'-\vec{q}\downarrow}) - n_{\vec{k}+\vec{q}\uparrow}n_{\vec{k}'-\vec{q}\downarrow}}{\epsilon_{\vec{k}} + \epsilon_{\vec{k}'} - \epsilon_{\vec{k}+\vec{q}} - \epsilon_{\vec{k}'-\vec{q}}}$$

$$f_{\sigma\sigma'}(\theta) = \frac{\tilde{U}}{2}\left[\left\{1 + \frac{\tilde{U}N(\epsilon_F)}{2}\left(2 + \frac{\cos\theta}{2\sin(\theta/2)}\ln\frac{1+\sin(\theta/2)}{1-\sin(\theta/2)}\right)\right\}\delta_{\sigma\sigma'} - \left\{1 + \frac{\tilde{U}N(\epsilon_F)}{2}\left(1 - \frac{\sin(\theta/2)}{2}\ln\frac{1+\sin(\theta/2)}{1-\sin(\theta/2)}\right)\right\}\sigma\sigma'\right]$$

Landau parameters:

$$f_{\sigma\sigma'}(\theta) = \frac{\tilde{U}}{2}\left[\left\{1 + \frac{\tilde{U}N(\epsilon_F)}{2}\left(2 + \frac{\cos\theta}{2\sin(\theta/2)}\ln\frac{1+\sin(\theta/2)}{1-\sin(\theta/2)}\right)\right\}\delta_{\sigma\sigma'} - \left\{1 + \frac{\tilde{U}N(\epsilon_F)}{2}\left(1 - \frac{\sin(\theta/2)}{2}\ln\frac{1+\sin(\theta/2)}{1-\sin(\theta/2)}\right)\right\}\sigma\sigma'\right]$$

$$f_{\sigma\sigma'}(\vec{k}, \vec{k}') = f^s(\hat{k}, \hat{k}') + \sigma\sigma' f^a(\hat{k}, \hat{k}')$$

$$f^{s,a}(\hat{k}, \hat{k}') = \sum_{l=0}^{\infty} f_l^{s,a} P_l(\cos\theta_{\hat{k},\hat{k}'})$$

$$N(\epsilon_F) f_l^{s,a} = F_l^{s,a}$$

Landau parameters:

$$f_{\sigma\sigma'}(\theta) = \frac{\tilde{U}}{2}\left[\left\{1 + \frac{\tilde{U}N(\epsilon_F)}{2}\left(2 + \frac{\cos\theta}{2\sin(\theta/2)}\ln\frac{1+\sin(\theta/2)}{1-\sin(\theta/2)}\right)\right\}\delta_{\sigma\sigma'}\right.$$

$$\left. - \left\{1 + \frac{\tilde{U}N(\epsilon_F)}{2}\left(1 - \frac{\sin(\theta/2)}{2}\ln\frac{1+\sin(\theta/2)}{1-\sin(\theta/2)}\right)\right\}\sigma\sigma'\right]$$

$$f_{\sigma\sigma'}(\vec{k}, \vec{k}') =$$

$$f^{s,a}(\hat{k}, \hat{k}') =$$

$$N(\epsilon_F) f_l^{s,a}$$

$\underline{\tilde{u} = \tilde{U}N(\epsilon_F) > 0}$ repulsive interaction

$$F_0^s = \tilde{u}\left\{1 + \tilde{u}\left(1 + \frac{1}{6}(2 + \ln 2)\right)\right\} = \tilde{u} + 1.449\,\tilde{u}^2 > 0$$

$$F_0^a = -\tilde{u}\left\{1 + \tilde{u}\left(1 - \frac{2}{3}(1 - \ln 2)\right)\right\} = -\tilde{u} - 0.895\,\tilde{u}^2 < 0$$

$$F_1^s = \tilde{u}^2\frac{2}{15}(7\ln 2 - 1) = 0.514\,\tilde{u}^2 > 0$$

$\underline{\tilde{u} = \tilde{U} N(\epsilon_F) > 0}$ repulsive interaction

$$F_0^s = \tilde{u} \left\{ 1 + \tilde{u} \left(1 + \frac{1}{6}(2 + \ln 2) \right) \right\} = \tilde{u} + 1.449\, \tilde{u}^2 > 0$$

$$F_0^a = -\tilde{u} \left\{ 1 + \tilde{u} \left(1 - \frac{2}{3}(1 - \ln 2) \right) \right\} = -\tilde{u} - 0.895\, \tilde{u}^2 < 0$$

$$F_1^s = \tilde{u}^2 \frac{2}{15}(7\ln 2 - 1) = 0.514\, \tilde{u}^2 > 0$$

$$\frac{\kappa}{\kappa_0} = \frac{m^*}{m} \frac{1}{1 + F_0^s} < 1$$

less compressible

$$\frac{\chi}{\chi_0} = \frac{m^*}{m} \frac{1}{1 + F_0^a} > 1$$

more spin polarizable

↓

ferromagnetism

$$\frac{m^*}{m} = 1 + \frac{1}{3} F_1^s > 1$$

higher effective mass

distribution function

ground state $$|\Psi\rangle = |\Psi^{(0)}\rangle + |\Psi^{(1)}\rangle + \cdots$$

$$|\Psi^{(0)}\rangle = |\Psi_0\rangle$$

$$|\Psi^{(1)}\rangle = \frac{U}{\Omega} \sum_{\vec{k},\vec{k}',\vec{q}} \sum_{s,s'} \frac{\hat{c}^\dagger_{\vec{k}+\vec{q},s} \hat{c}^\dagger_{\vec{k}'-\vec{q},s'} \hat{c}_{\vec{k}',s'} \hat{c}_{\vec{k},s}}{\epsilon_{\vec{k}} + \epsilon_{\vec{k}'} - \epsilon_{\vec{k}+\vec{q}} - \epsilon_{\vec{k}'-\vec{q}}} |\Psi_0\rangle$$

$$\langle \hat{n}_{\vec{k}s} \rangle = \frac{\langle \Psi | \hat{c}^\dagger_{\vec{k}s} \hat{c}_{\vec{k}s} | \Psi \rangle}{\langle \Psi | \Psi \rangle} = \frac{\langle \Psi^{(0)} | \hat{c}^\dagger_{\vec{k}s} \hat{c}_{\vec{k}s} | \Psi^{(0)} \rangle + \langle \Psi^{(1)} | \hat{c}^\dagger_{\vec{k}s} \hat{c}_{\vec{k}s} | \Psi^{(1)} \rangle}{\langle \Psi | \Psi \rangle}$$

$$\langle \hat{n}_{\vec{k}s} \rangle = \langle \hat{n}_{\vec{k}s} \rangle^{(0)} + \langle \hat{n}_{\vec{k}s} \rangle^{(2)} + \cdots$$

distribution function

$$\langle \hat{n}_{\vec{k}s} \rangle = \langle \hat{n}_{\vec{k}s} \rangle^{(0)} + \langle \hat{n}_{\vec{k}s} \rangle^{(2)} + \cdots \qquad \langle \hat{n}_{\vec{k}s} \rangle^{(0)} = \Theta(k_F - |\vec{k}|)$$

$$\langle \hat{n}_{\vec{k}s} \rangle^{(2)} = \begin{cases} -\dfrac{U^2}{\Omega^2} \displaystyle\sum_{\vec{k}_1, \vec{k}_2, \vec{k}_3} \dfrac{(1 - n_{\vec{k}_1})(1 - n_{\vec{k}_2}) n_{\vec{k}_3}}{(\epsilon_{\vec{k}} + \epsilon_{\vec{k}_3} - \epsilon_{\vec{k}_1} - \epsilon_{\vec{k}_2})^2} \delta_{\vec{k}+\vec{k}_3, \vec{k}_1+\vec{k}_2} & |\vec{k}| < k_F \\ \dfrac{U^2}{\Omega^2} \displaystyle\sum_{\vec{k}_1, \vec{k}_2, \vec{k}_3} \dfrac{n_{\vec{k}_1} n_{\vec{k}_2} (1 - n_{\vec{k}_3})}{(\epsilon_{\vec{k}_1} + \epsilon_{\vec{k}_2} - \epsilon_{\vec{k}} - \epsilon_{\vec{k}_3})^2} \delta_{\vec{k}+\vec{k}_3, \vec{k}_1+\vec{k}_2} & |\vec{k}| > k_F \end{cases}$$

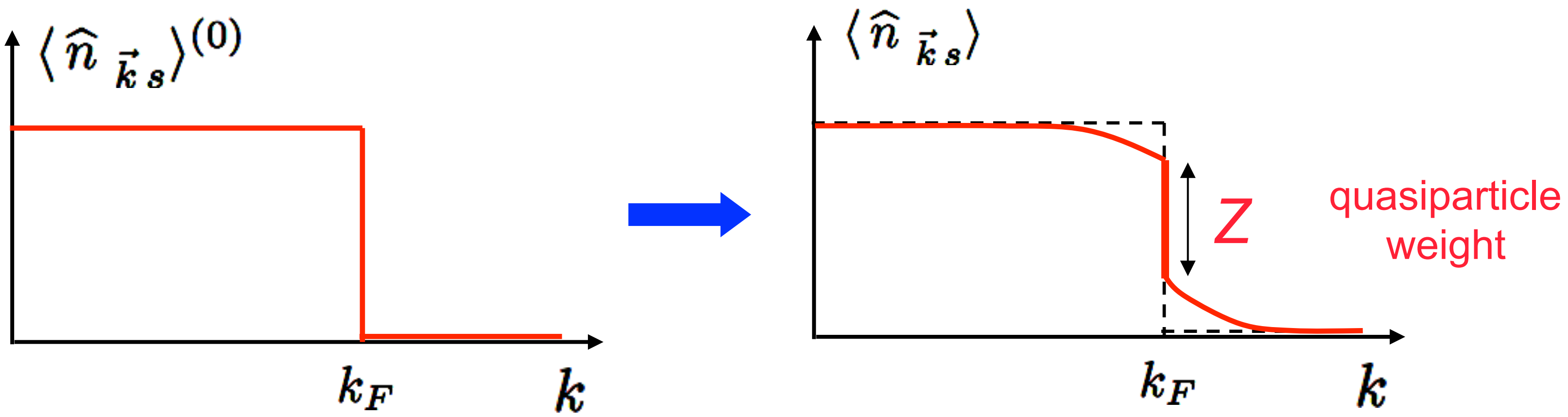
3-dimensional Fermigas

$$\langle \hat{n}_{\vec{k}_{F-}} \rangle - \langle \hat{n}_{\vec{k}_{F+}} \rangle = 1 - \left(\frac{U N(\epsilon_F)}{2} \right)^2 \ln 2$$

distribution function

$$\langle \hat{n}_{\vec{k}s} \rangle = \langle \hat{n}_{\vec{k}s} \rangle^{(0)} + \langle \hat{n}_{\vec{k}s} \rangle^{(2)} + \cdots \qquad \langle \hat{n}_{\vec{k}s} \rangle^{(0)} = \Theta(k_F - |\vec{k}|)$$

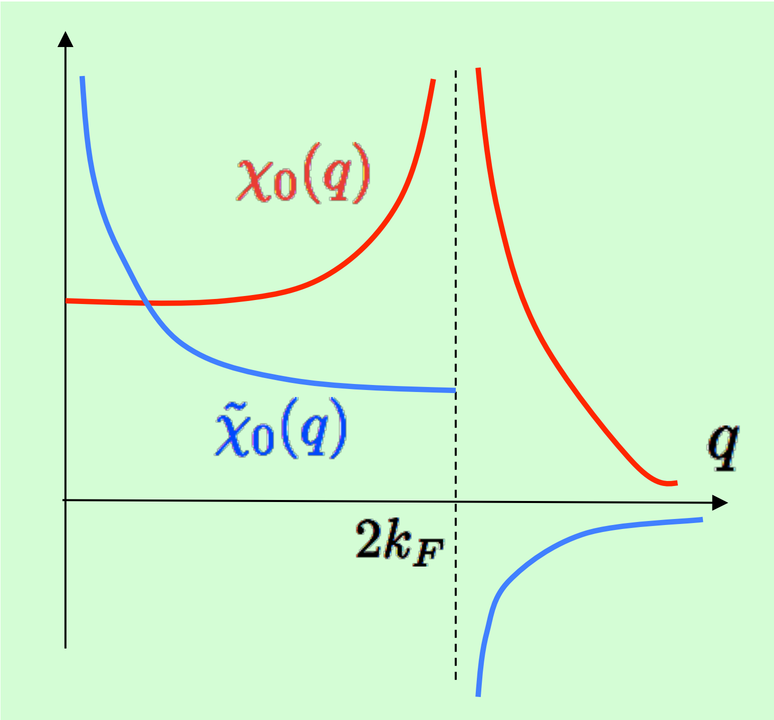
$$\langle \hat{n}_{\vec{k}_{F-}} \rangle - \langle \hat{n}_{\vec{k}_{F+}} \rangle = 1 - \left(\frac{UN(\epsilon_F)}{2} \right)^2 \ln 2$$



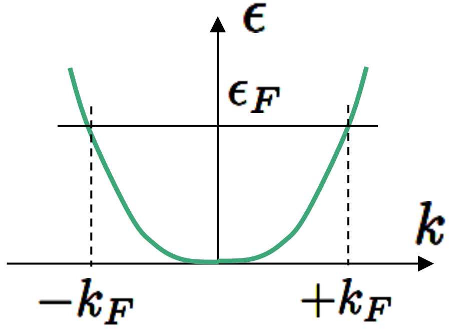
Landau parameters

$$f_{\sigma\sigma}(k_F, k'_F) = \frac{\tilde{U}^2}{2}\chi_0(k_F - k'_F)$$

$$f_{\sigma\bar{\sigma}}(k_F, k'_F) = \tilde{U} - \frac{\tilde{U}^2}{2}\{2\tilde{\chi}_0(k_F + k'_F) - \chi_0(k_F - k'_F)\}$$



$$q = 0, \pm 2k_F$$



$$f_{\sigma\sigma}(\pm k_F, \mp k_F) = \infty$$

$$f_{\sigma\bar{\sigma}}(k_F, \mp k_F) = \infty$$

breakdown of perturbation approach !

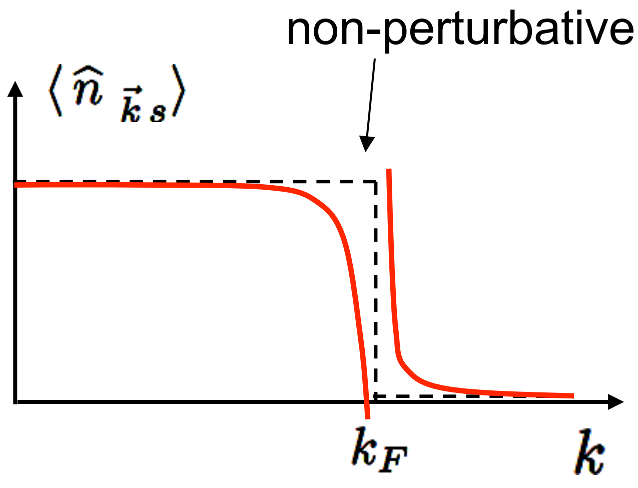
distribution function

$$\langle \hat{n}_{\vec{k}s} \rangle = \langle \hat{n}_{\vec{k}s} \rangle^{(0)} + \langle \hat{n}_{\vec{k}s} \rangle^{(2)} + \cdots \qquad \langle \hat{n}_{\vec{k}s} \rangle^{(0)} = \Theta(k_F - |\vec{k}|)$$

$$\langle \hat{n}_{\vec{k}s} \rangle^{(2)} = \begin{cases} -\frac{U^2}{\Omega^2} \sum_{\vec{k}_1,\vec{k}_2,\vec{k}_3} \frac{(1-n_{\vec{k}_1})(1-n_{\vec{k}_2})n_{\vec{k}_3}}{(\epsilon_{\vec{k}} + \epsilon_{\vec{k}_3} - \epsilon_{\vec{k}_1} - \epsilon_{\vec{k}_2})^2} \delta_{\vec{k}+\vec{k}_3,\vec{k}_1+\vec{k}_2} & |\vec{k}| < k_F \\ \frac{U^2}{\Omega^2} \sum_{\vec{k}_1,\vec{k}_2,\vec{k}_3} \frac{n_{\vec{k}_1} n_{\vec{k}_2}(1-n_{\vec{k}_3})}{(\epsilon_{\vec{k}_1} + \epsilon_{\vec{k}_2} - \epsilon_{\vec{k}} - \epsilon_{\vec{k}_3})^2} \delta_{\vec{k}+\vec{k}_3,\vec{k}_1+\vec{k}_2} & |\vec{k}| > k_F \end{cases}$$

1-dimensional Fermigas

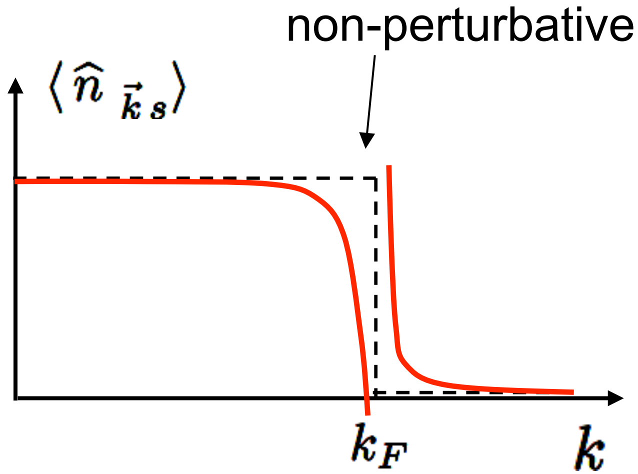
$$\langle \hat{n}_{\vec{k}s} \rangle^{(2)} \approx \begin{cases} \frac{1}{8\pi^2} \frac{U^2}{\hbar^2 v_F^2} \ln \frac{k_+}{k - k_F} & k > k_F \\ -\frac{1}{8\pi^2} \frac{U^2}{\hbar^2 v_F^2} \ln \frac{k_-}{k_F - k} & k < k_F \end{cases}$$



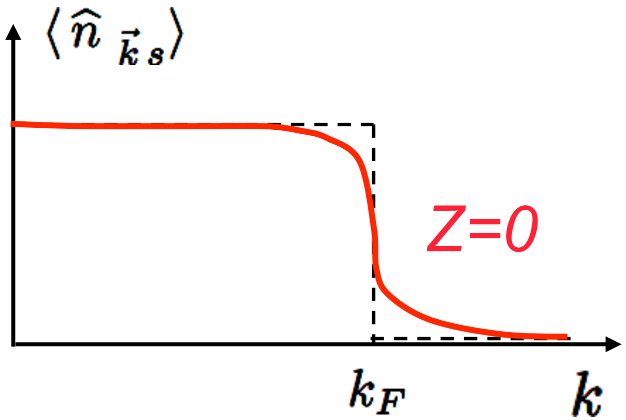
distribution function

$$\langle \hat{n}_{\vec{k}\,s} \rangle = \langle \hat{n}_{\vec{k}\,s} \rangle^{(0)} + \langle \hat{n}_{\vec{k}\,s} \rangle^{(2)} + \cdots \qquad \langle \hat{n}_{\vec{k}\,s} \rangle^{(0)} = \Theta(k_F - |\vec{k}\,|)$$

$$\langle \hat{n}_{\vec{k}\,s} \rangle^{(2)} \approx \begin{cases} \dfrac{1}{8\pi^2} \dfrac{U^2}{\hbar^2 v_F^2} \ln \dfrac{k_+}{k - k_F} & k > k_F \\ -\dfrac{1}{8\pi^2} \dfrac{U^2}{\hbar^2 v_F^2} \ln \dfrac{k_-}{k_F - k} & k < k_F \end{cases}$$



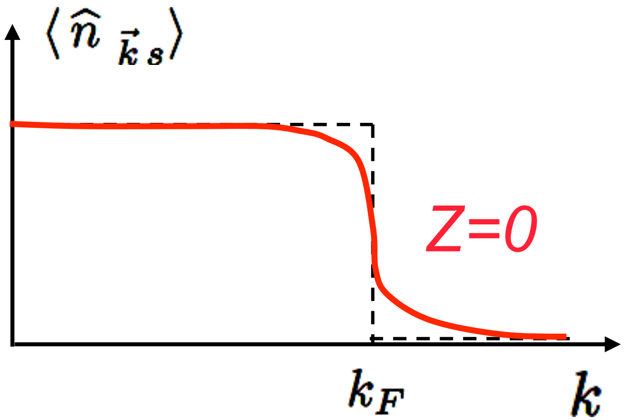
1-dimensional Fermigas



no quasiparticle weight
⬇
not a Fermi liquid

Tomonaga Luttinger liquid

Tomonaga-Luttinger liquid



no quasiparticles

Fermi liquid behavior disappears

excitations:

collective modes (bosonization of Fermions)

separation of charge and spin excitations