Exercise 1. ***Specific Heat of a Metal, a Semiconductor, and Graphene***

The specific heat at constant volume is defined as

$$c_V = \frac{1}{V}\left(\frac{\partial U}{\partial T}\right)_{V,N} \tag{1}$$

where U is the internal energy of the system.

a) Calculate the specific heat of a metal for small temperatures $k_\mathrm{B}T \ll \mu$. Neglect the Coulomb interaction and assume there is only one conduction band with a parabolic dispersion relation

$$\varepsilon_\mathbf{k} = \frac{\hbar^2\mathbf{k^2}}{2m}. \tag{2}$$

Note that in order to have a fixed number of particles, the chemical potential will shift when temperature is changed! You will have to use the Sommerfeld expansion

$$\int_{-\infty}^{\infty} \mathrm{d}\varepsilon\ \frac{H(\varepsilon)}{\mathrm{e}^{(\varepsilon-\mu)/k_BT}+1} = \int_{-\infty}^{\mu} \mathrm{d}\varepsilon\ H(\varepsilon) + \sum_{n=1}^{\infty} a_n \left(k_BT\right)^{2n} \left.\frac{\mathrm{d}^{2n-1}H(\varepsilon)}{\mathrm{d}\varepsilon^{2n-1}}\right|_{\varepsilon=\mu} \tag{3}$$

where $a_1 = \pi^2/6$, $a_2 = 7\pi^4/360, \ldots$.

b) Calculate the specific heat of an undoped semiconductor under the assumption $k_\mathrm{B}T \ll E_g$, where E_g is the band gap. Show that it contains an ideal gas-like contribution $(3/2)n(T)k_B$ where $n(T)$ is the number of excitations, and a correction. Is this correction small or large?

To proceed, use the $\mathbf{k}\cdot\mathbf{p}$-approximation for the band spectrum, given by eq. (2.27) in the lecture notes. The chemical potential $\mu(T)$ has to be calculated from the condition, that the number of electrons in the conduction band ($n_e(T)$) is equal to the number of holes in the valence band ($n_h(T)$).

c) Calculate the specific heat of graphene at half filling. (For the band structure of graphene cf. previous exercise sheet.) Note that the perfect particle-hole symmetry fixes chemical potential to the Dirac nodes at all temperatures. To simplify the calculation, approximate the dispersion around the two Dirac points as

$$\varepsilon_\mathbf{k} = \pm\hbar v_\mathrm{F}|\mathbf{k}|, \tag{4}$$

where $\mathbf{k}$ is relative to the position of a Dirac node.

Exercise 2. ***Spin Susceptibility of a Metal, a Semiconductor, and Graphene***

The Pauli spin susceptibility is defined as

$$\chi_\mathrm{Pauli} = \left(\left.\frac{\partial M}{\partial H}\right|_{H=0}\right)_{T,V,N} \tag{5}$$

where M is the net electron magnetization.

a) Calculate the Pauli spin susceptibility of a metal due to its conduction electrons. Assume that the magnetic field couples only to the electron spin.

b) Calculate the spin susceptibility of a semiconductor with gap $E_g > 0$ and compare the result to an ideal paramagnetic gas.

c) Calculate the spin susceptibility of graphene

Use the same assumptions about the electron spectra as in Exercise 1.

Exercise 3. ***Compressibility of a Metal, a Semiconductor, and Graphene***

The compressibility of a system with volume V is defined as

$$\kappa = -\frac{1}{V}\left(\frac{\partial V}{\partial p}\right)_{T,N} = \frac{1}{n^2}\left(\frac{\partial n}{\partial \mu}\right)_{T,V}. \tag{6}$$

a) Show that the two formulas above are equivalent. To do so, you may find useful the Gibbs-Duhem relation

$$N\mathrm{d}\mu = V\mathrm{d}p - S\mathrm{d}T \tag{7}$$

which states that the equilibrium parameters μ, p and T cannot be varied independently.

b) Calculate the compressibility of a gas of free and independent electrons with density n. (Coulomb interaction is neglected.)

c) Calculate compressibility of an electron gas in the jellium model where Coulomb interaction are considered. Use that within the Hartree-Fock theory (cf. section 3.1.2 in the lecture notes), the energy per electron is given by

$$\varepsilon_g = \left(\frac{2.21}{r_s^2} - \frac{0.916}{r_s}\right)\mathrm{Ry} \tag{8}$$

where $1\mathrm{Ry} = e^2/2a_\mathrm{B}$ and $r_s = d/a_\mathrm{B}$ with a_B the Bohr radius and d the radius of the average volume occupied by one electron

$$a_\mathrm{B} = \frac{\hbar^2}{me^2} \qquad n = \frac{1}{\frac{4}{3}\pi d^3}, \tag{9}$$

$n = k_\mathrm{F}^3/3\pi^2$ is the number of electrons per unit volume. Compare results (a) and (b).

d) Show that electrons in an insulator are incompressible.

e) Calculate compressibility of the delocalized p_z-electrons in graphene.