Problem Set 7: Scattering amplitudes in gauge theories

Discussion on Wednesday 14.05 13:45-14:30, HIT H 51
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Exercise 12 – 2-point tensor integral reduction

In class we discussed the strategy to reduce a 4-point and a 3-point tensor integral. Apply this stategy to the 2-point tensor integrals

$$I_{2,r} = \int \frac{d^D l}{(2\pi)^D} \frac{\mathcal{N}_{2,r}(l)}{d_1 d_2}$$

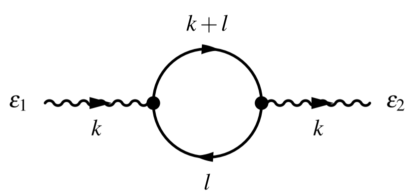
and show that one may always reduce the numerator polynomial to the form

$$\begin{aligned}\mathcal{N}_{2,r\leq 2}(l) = \tilde{b}_0 &+ \tilde{b}_1\,(l\cdot n_2) + \tilde{b}_2\,(l\cdot n_3) + \tilde{b}_3\,(l\cdot n_4) + \tilde{b}_4\,(l\cdot n_\epsilon)^2 \\ &+ \tilde{b}_5\,[\,(l\cdot n_2)^2 - (l\cdot n_4)^2\,] + \tilde{b}_6\,[\,(l\cdot n_3)^2 - (l\cdot n_4)^2\,] \\ &+ \tilde{b}_7\,(l\cdot n_2)(l\cdot n_3) + \tilde{b}_8\,(l\cdot n_2)(l\cdot n_4) + \tilde{b}_9\,(l\cdot n_3)(l\cdot n_4)\,.\end{aligned}$$

modulo 1-point integrals, i.e. terms proportional to d_1 or d_2.

Exercise 13 – Photon self-energy in QED

We now wish to apply the reduction technique to the calculation of the one-loop correction to the photon propagator in massless QED:



$$\Pi_{12} = -e^2 \int \frac{d^D l}{(2\pi)^D} \frac{\mathrm{Tr}(\not\epsilon_1(\not k + \not l)\not\epsilon_2\,\not l\,)}{l^2(l+k)^2}\,.$$

Here we have introduced polarization vectors ϵ_i at the external legs which are to be understood as placeholders. Nethertheless transversality demands the vanishing of Π_{12} for $\epsilon_i \to k$. For the reduction of this integral note that the tadpoles (1-point functions) all vanish in dimensional regularization due to masslessness.
In order to isolate the bubble coefficients of Π_{12} we put the two fermion propagators on-shell:

$$l^2 = 0\,, \qquad (l+k)^2 = 0\,.$$

Show that these two equations are solved by

$$l^\mu = -\tfrac{1}{2}k^\mu + l_\perp^\mu + (l\cdot n_\epsilon)\,n_\epsilon^\mu \qquad \text{obeying} \quad l_\perp^2 + (l\cdot n_\epsilon)^2 = -\tfrac{1}{4}\,k^2\,.$$

Now to insert these on-shell l^μ found into the numerator of $\mathit{\Pi}_{12}$. This will directly yield the bubble coefficients as any off-shell components of l^μ would only give rise to terms proportional to l^2 or $(l+k)^2$ in the numerator which contribute to the vanishing tadpole integrals. Work this out and show that the self-energy takes the form

$$\begin{aligned}\mathit{\Pi}_{12} = &-\frac{4}{3}\, e^2\, k^2 \left(\epsilon_1 \cdot \epsilon_2 - \frac{(\epsilon_1 \cdot k)\,(\epsilon_2 \cdot k)}{k^2}\right) \int \frac{d^D l}{(2\pi)^D}\, \frac{1}{l^2\,(l+k)^2} \\ &+ \frac{8}{3}\, e^2 \left(\epsilon_1 \cdot \epsilon_2 - \frac{(\epsilon_1 \cdot k)\,(\epsilon_2 \cdot k)}{k^2}\right) \int \frac{d^D l}{(2\pi)^D}\, \frac{(l \cdot n_\epsilon)^2}{l^2\,(l+k)^2}\,,\end{aligned}$$

in fact recovering transversality.