Exercise 1. ***Canonical Quantization***

Consider the Lagrangian for a free real scalar field $\phi(x)$:

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2$$

(a) Using the Euler-Lagrange equation

$$\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\phi\right)}\right) - \frac{\partial\mathcal{L}}{\partial\phi} = 0$$

compute the equation of motion for the real scalar field.

(b) Use canonical quantization to derive the Feynman propagator $D_F(x-y) = \langle 0|T\phi(x)\phi(y)|0\rangle$.

Hint. First compute the Hamiltonian of the system and apply the commutation relations. Then rewrite the field ϕ in terms of creation and annihilation operators.

Exercise 2. ***Path Integral Quantization***

(a) Evaluate the generating path integral for the free real scalar field working in Fourier space and following the analogous derivation of the simple harmonic oscillator done in the lecture.

(b) Find the Fourier representation of the propagator.

(c) Integrate the Fourier representation of the propagator over the energy using the Cauchy theorem. Pay attention to the conditions on the time variable in order to be able to use a closed contour of integration.

Exercise 3. ***Anharmonic oscillator***

As you have seen, while for the free theory the path integral is usually gaussian and hence exactly solvable, for an interacting theory things are far more complicated and one method to compute the integral is via a perturbative expansion in the coupling constant (if they are small enough) around the free theory.

Consider the anharmonic oscillator whose action is given by

$$S[x] = \int dt\,\left(\frac{1}{2}\dot{x}^2 - \frac{\omega^2}{2}x^2 - \frac{g}{3!}x^3 - \frac{\lambda}{4!}x^4\right) \qquad (1)$$

and assume that the coupling g, λ are sufficiently small to be able to do a good perturbative expansion.

(a) Write the generating function $Z[J]$ for the system, in the presence of an external source, in terms of the one of the free theory $Z_0[J]$.

Hint. Split the action in the sum of two parts, the free part and the interacting one.

(b) Consider the Euclidean version of the theory, where the calculations are easier to compute. Consider the limit $\beta \to \infty$, so that only the ground state actually contributes to Z_E, i.e.

$$\begin{aligned} Z_E[0] &= \langle 1 \rangle_U = \lim_{\beta\to\infty} \langle 0|e^{-\beta\hat{H}}|0\rangle = \lim_{\beta\to\infty} e^{-\beta E_0} \\ &= \langle e^{-S_{E,int}[x]} \rangle_{0,U} = \lim_{\beta\to\infty} e^{-\beta(E_0^{(0)}+\Delta E_0)} \end{aligned} \tag{2}$$

where the subscript $0, U$ mean unnormalised averaging (U) with the free theory (0), E_0 is the exact energy of the ground state $|0\rangle$, while $E_0^{(0)}$ is the ground state energy of the simple harmonic oscillator and ΔE_0 is the difference between the two.

Compute perturbatively the first non-vanishing corrections for the two cases

1) $g = 0,\ \lambda \neq 0$
2) $g \neq 0,\ \lambda = 0$

Hint. First remember that in this case the action is given by

$$S_E[x] = \lim_{\beta\to\infty} \int_{-\beta/2}^{\beta/2} d\tau \left(\frac{1}{2}\dot{x}^2 + \frac{\omega^2}{2}x^2 + \frac{g}{3!}x^3 + \frac{\lambda}{4!}x^4 \right)$$

Expand $\langle e^{-S_{E,int}[x]} \rangle_{0,U}$ *in series of the coupling constant of interest and collect the expansion in terms of* $\langle 1 \rangle_{U,0}$*, so as to have the normalised correlation functions of the free theory to calculate using Wick's Theorem. Notice that you have already calculated the 2-point function for the simple harmonic oscillator in the previous exercise sheet!*