Exercise 1. *Free particle propagator*

(a) Calculate the following integral

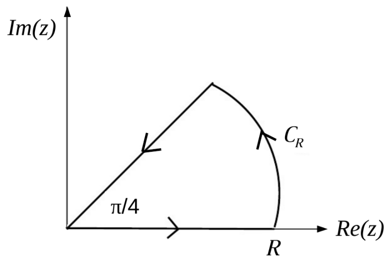
$$\int_{-\infty}^{\infty} dx\ e^{-ax^2} = \sqrt{\frac{\pi}{a}}$$

Hint. Square the left hand side and use polar coordinates.

(b) Prove the Fresnel integral relation:

$$\int_{-\infty}^{\infty} dx\ e^{iax^2+ikx} = \sqrt{\frac{i\pi}{a}} e^{-ik^2/4a}, \qquad (\mathrm{Im}\ a > 0)$$

Hint. First compute $\int_{-\infty}^{\infty} dx\ e^{iax^2}$ *by considering the integral* $\oint_{C_R} dz\ e^{iaz^2}$ *in the complex plane around the contour given below.*



(c) Show explicitly that the free particle propagator

$$K(x, x'; t-t') = \left(\frac{m}{2\pi i\hbar(t-t')}\right)^{1/2} e^{\frac{im(x-x')^2}{2\hbar(t-t')}}$$

satisfies the following completeness relation:

$$K(x_b, x_a; t_b - t_a) = \int_{-\infty}^{\infty} \mathrm{d}x\ \langle x_b, t_b | x, t\rangle \langle x, t | x_a, t_a\rangle, \qquad \forall t \in (t_a, t_b)$$

Hint. Use the integral relation in part (b).

Exercise 2. *Free particle wave function*

Consider a free particle at $t = 0$ (i.e. a wave function $\propto e^{ipx/\hbar}$). Calculate its time evolution using the propagator, i.e.

$$\psi(x,t) = \int dx'\ K(x, x'; t)\ \psi(x', t' = 0)$$

Hint. Use the Fresnel integral relation in part (b) of exercise 1.

Exercise 3. ***A bit more on propagators***

Using

$$K(x, x'; t - t') = \sum_{\beta} e^{-\frac{i}{\hbar} E_\beta (t-t')} \langle x|\beta\rangle\langle\beta|x'\rangle$$

Compute the propagator for

(a) a free particle

(b) the simple harmonic oscillator. Why is this computation more difficult than the previous one? Probably we need a different approach. Let's think about it!

Exercise 4. ***Explicit propagator calculation***

Given a Lagrangian of the form:

$$L = \frac{1}{2} f(x)\dot{x}^2 + g(x)\dot{x} - V(x)$$

(a) Calculate the Hamiltonian.

(b) Calculate the propagator for some small time interval δt.

(c) Determine the path integral expression for the propagator at large times, and show that the measure of the path integration is modified by a factor $\sqrt{f(x)}$ compared with the measure in the free particle case.