Exercise 1. ***1-loop correction to the Effective Potential***

We have seen in the lecture that we can reorganize the perturbative expansion of the generating functional to obtain an expansion including only tree diagrams, which yet includes all the information about the theory. The action of this new expansion is the Quantum Effective Action. Therefore it is important to know how to compute this action and the corresponding potential. Recall that the generating functional is:

$$Z[J] = \int \mathcal{D}\phi \, exp\Big\{ i \int d^4x \, (\mathcal{L} + J\phi(x)) \Big\} \tag{1}$$

The main contribution to this integral is from $\phi_{cl}(x)$ and therefore we can make an expansion around this classical value:

$$\phi(x) = \phi_{cl}(x) + \eta(x) \tag{2}$$

where $\eta(x)$ is the fluctuation. In Equation (1) we now have:

$$\begin{aligned} \int d^4x \, (\mathcal{L} + J\phi) &= \int d^4x \, (\mathcal{L}[\phi_{cl}] + J\phi_{cl}) + \int d^4x \, \eta(x) \, \left(\frac{\delta \mathcal{L}}{\delta \phi}|_{\phi=\phi_{cl}} + J \right) \\ &+ \frac{1}{2} \int d^4x \, d^4y \, \eta(x)\eta(y) \frac{\delta^2 \mathcal{L}}{\delta\phi(x)\delta\phi(y)}|_{\phi=\phi_{cl}} \\ &+ \frac{1}{3!} \int d^4x \, d^4y \, d^4z \, \eta(x)\eta(y)\eta(z) \frac{\delta^3 \mathcal{L}}{\delta\phi(x)\delta\phi(y)\delta\phi(z)}|_{\phi=\phi_{cl}} + \cdots \end{aligned} \tag{3}$$

Note that the terms linear in η vanishes due to equation of motion. Therefore up to quadratic order in η we have:

$$\int d^4x \, (\mathcal{L} + J\phi) = \int d^4x \, (\mathcal{L}[\phi_{cl}] + J\phi_{cl}) + \frac{1}{2} \int d^4x \, d^4y \, \eta(x) \, \eta(y) \, \frac{\delta^2 \mathcal{L}}{\delta\phi(x)\delta\phi(y)}|_{\phi=\phi_{cl}} \tag{4}$$

In the path integral we now have an integral over $\eta(x)$ since $\mathcal{D}\phi = \mathcal{D}\eta$. Let us look at the form of this explicitly:

$$\int \mathcal{D}\eta \, exp\Big\{ i \int d^4x \, (\mathcal{L}[\phi_{cl}] + J\phi_{cl}) + \frac{1}{2} \int d^4x \, d^4y \, \eta(x) \, \frac{\delta^2 \mathcal{L}}{\delta\phi(x)\delta\phi(y)} \eta(y)|_{\phi=\phi_{cl}} \Big\} \tag{5}$$

The quadratic term in η is indeed the inverse of the propagator for it:

$$\tilde{D}(p) = -i \int d^4x \, d^4y \, \left(\frac{\delta^2 \mathcal{L}}{\delta\phi^2} \right)^{-1}_{\phi=\phi_{cl}} \tag{6}$$

The path integral over η is Gaussian and when computed explicitly it gives the functional determinant. Therefore Equation (5) becomes:

$$exp\Big\{ i \int d^4x \, (\mathcal{L}[\phi_{cl}] + J\phi_{cl}) \Big\} \, \left(\det \left(-\frac{\delta^2 \mathcal{L}}{\delta\phi\delta\phi} \right) \right)^{-1/2}_{\phi=\phi_{cl}} \tag{7}$$

The determinant is indeed:

$$\left(\det \left(-\frac{\delta^2 \mathcal{L}}{\delta\phi^2} \right) \right)^{-1/2}_{\phi=\phi_{cl}} = exp\Big\{ -\frac{1}{2} \mathrm{Tr} \ln \left(-\frac{\delta^2 \mathcal{L}}{\delta\phi^2} \right) \Big\}_{\phi=\phi_{cl}} \tag{8}$$

Combining everything the exponent in the path integral becomes:

$$-iW[J] = i\int d^4x\ (\mathcal{L}[\phi_{cl}] + J\phi_{cl}) - \frac{1}{2}\mathrm{Tr}\ln\left(-\frac{\delta^2\mathcal{L}}{\delta\phi\delta\phi}\right)_{\phi=\phi_{cl}} \tag{9}$$

The quantum effective action is defined as the Legendre transform:

$$\Gamma[\phi_{cl}] = W[J] - \int d^4x J(x)\phi_{cl} \tag{10}$$

$$= \int d^4x\,\mathcal{L}[\phi_{cl}] + \frac{i}{2}\mathrm{Tr}\ln\left(-\frac{\delta^2\mathcal{L}}{\delta\phi\delta\phi}\right)_{\phi=\phi_{cl}} \tag{11}$$

Now consider the action of a scalar field:

$$S[\phi] = \int d^4x\left(\frac{1}{2}\left(\partial^\mu\phi^i\right)^2 + V(\phi)\right), \quad V(\phi^i) = -\frac{1}{2}m^2(\phi^i)^2 - \frac{\lambda}{4!}((\phi^i)^2)^2 \tag{12}$$

The effective action up to one loop is then

$$\Gamma[\phi_{cl}] = \int d^4x\left(\frac{1}{2}\left(\partial^\mu\phi^i_{cl}\right)^2 + V(\phi_{cl})\right) + \frac{1}{2}\mathrm{Tr}\ \ln\left(\partial^2 + V''(\phi_{cl})\right) \tag{13}$$

The Effective potential up to one loop is

$$V_{eff}(\phi_{cl}) = -\frac{1}{VT}\Gamma[\phi_{cl}] \tag{14}$$

$$V_{eff}(\phi_{cl}) = V(\phi_{cl}) - \frac{1}{VT}\frac{1}{2}\mathrm{Tr}\ \ln\left(\partial^2 + V''(\phi_{cl})\right) \tag{15}$$

(a) Calculate the second term corresponding to the one loop correction. For this go to momentum space. You should find:

$$V^{(1)} = i\sum_{n=1}^{\infty}\frac{1}{2n}\int\frac{d^dk}{(2\pi)^d}\left(\frac{\lambda\phi^2}{2}\right)^n\frac{1}{(k^2-m^2)^n} \tag{16}$$

(b) Integrate the loop momentum k and introduce the Pochhammer symbol

$$(\alpha,\beta) = \Gamma(\alpha+\beta)/\Gamma(\alpha) \tag{17}$$

to recast the above expression into the form:

$$V^{(1)} = -\frac{1}{2}\left(\frac{m^2}{4\pi}\right)^{d/2}\Gamma(-d/2)\left(\sum_{n=1}^{\infty}\frac{(-d/2,n)}{n!}w^n\right) \tag{18}$$

where w has to be determined. Then use the relation $\sum_{n=0}^{\infty}\frac{(-d/2,n)}{n!}w^n = (1-w)^{d/2}$ (Note that the sum strats from $n=0$ in this case.). Substitute $d = 4-2\epsilon$ and expand in ϵ.

Exercise 2. ***The effective action Slavnov-Taylor identity***

The BRST symmetry plays an important role in determining the dynamics of gauge fixed field theories. In this exercise we will show that this symmetry implies non-trivial relations between connected Green's functions – the *effective action Slavnov-Taylor identity.*

Given a non-abelian gauge theory with gauge fields A_μ^a and scalar matter fields φ^i (with representation index i), the BRST transformation δ_B of the various fields can be defined as follows:

$$\begin{aligned}
\delta_B A_\mu^a &= -\frac{\theta}{g}\,(D_\mu \eta)^a \\
\delta_B \varphi^i &= -ig\theta\eta^a (T^a)^i{}_j \varphi^j \\
\delta_B w^a &= 0 \\
\delta_B \eta^a &= \frac{\theta}{2} f^{abc}\eta^b\eta^c \\
\delta_B \overline{\eta}^a &= \frac{\theta}{g} w^a
\end{aligned}$$

where w^a, η^a and $\overline{\eta}^a$ are auxilliary, ghost, and anti-ghost fields respectively.

(a) The BRST variation of a generic operator $\mathcal{O}$ is defined by:

$$\delta_B \mathcal{O} := [iQ_B, \mathcal{O}]_\pm$$

where the bracket is: $[\cdot,\cdot]_- := [\cdot,\cdot]$ if $\mathcal{O}$ contains an *even* number of anti-commuting variables, or: $[\cdot,\cdot]_+ := \{\cdot,\cdot\}$ if $\mathcal{O}$ contains an *odd* number of anti-commuting variables, and Q_B is the BRST charge discussed in Exercise 3 of Sheet 7. Given the subsidiary condition: $Q_B|\text{phys}\rangle = 0$ and the fact that Q_B is Hermitian, show that:

$$\langle 0|\delta_B \mathcal{O}|0\rangle = 0$$

(b) Let $\mathcal{O} = \mathrm{T}\left\{e^{i\mathcal{S}[J_I,K_I]}\right\}$, where T denotes time ordering and:

$$\begin{aligned}
\mathcal{S}[J_I,K_I] := \int d^4x\, \Big[& J^{\mu a}A_\mu^a + J_\varphi^i\varphi^i + J_\eta^a\eta^a + J_{\overline{\eta}}^a\overline{\eta}^a + J_w^a w^a \\
& - K^{\mu a}\frac{\theta}{g}(D_\mu\eta)^a - igK_\varphi^i\theta\eta^a(T^a)^i{}_j\varphi^j + \tfrac{1}{2}K_\eta^a\theta f^{abc}\eta^b\eta^c\Big]
\end{aligned}$$

where $J_\mu, J_w, J_\varphi^i, K_\mu, K_\varphi^i$ are complex number sources which are invariant under BRST transformations and $J_\eta, J_{\overline{\eta}}, K_\eta$ are anti-commuting complex number sources which are invariant under BRST transformations. Use the result of part (a) and the fact that: $\delta_B\left(e^{i\mathcal{S}}\right) = (i\delta_B\mathcal{S})\,e^{i\mathcal{S}}$ to show that:

$$i\int d^4x\, \langle 0|\mathrm{T}\left\{\left[-J^{\mu a}\tfrac{\theta}{g}(D_\mu\eta)^a - igJ_\varphi^i\theta\eta^a(T^a)^i{}_j\varphi^j + \tfrac{1}{2}J_\eta^a\theta f^{abc}\eta^b\eta^c + J_{\overline{\eta}}^a\tfrac{\theta}{g}w^a\right]e^{i\mathcal{S}[J,K]}\right\}|0\rangle = 0$$

Hint. Use the condition $\delta_B\delta_B F = 0$ for any field F.

(c) In the operator formalism we can write the time-ordered vacuum expectation value of an operator F as:

$$\langle \mathrm{T}\{F\}\rangle = \frac{\int \mathcal{D}\phi\, F[\phi]\, e^{iS[\phi]}}{\int \mathcal{D}\phi\, e^{iS[\phi]}} \tag{19}$$

where here $S[\phi]$ is the action of the theory considered. Consider now the generating functional:

$$Z[J_I, K_I] = e^{i\,W[J_I,K_I]} = N \int \mathcal{D}\phi_I \left(e^{i\int d^4x (K_I B_I + J_I \phi_I)} \right) e^{iS[\phi_I]} \tag{20}$$

where N is the usual normalisation factor:

$$N = \left(\int \mathcal{D}\phi_I \, e^{iS[\phi_I]} \right)^{-1}$$

If we compare this expression to (19) and if we define $|0\rangle$ to be the vacuum of the full theory, interactions included, we can actually rewrite it as:

$$e^{iW[J_I,K_I]} = \langle 0 | \mathrm{T} \left\{ e^{i\mathcal{S}[J_I,K_I]} \right\} |0\rangle \tag{21}$$

Then, the effective action Γ is given by:

$$\Gamma[\Psi_I, K_I] := W[J_I, K_I] - \sum_I J_I \Psi_I$$

$$\Psi_I := \frac{\delta W[J_I, K_I]}{\delta J_I}$$

where K_I and Ψ_I are treated as independent variables (with I labelling the various different fields), but $J = J[K_I, \Psi_I]$. By functionally differentiating (on the left) the expression for Γ above, show that:

$$\frac{\delta \Gamma}{\delta \Psi_I} = \begin{cases} -J_I, & \text{for bosonic fields} \\ +J_I, & \text{for fermionic fields} \end{cases}$$

Use this relation, the definitions above, and the expression derived in part (c) to derive the effective action Slavnov-Taylor identity:

$$i \int d^4x \left[\frac{\delta \Gamma}{\delta K^{\mu a}} \frac{\delta \Gamma}{\delta A^a_\mu} + \frac{\delta \Gamma}{\delta K^i_\varphi} \frac{\delta \Gamma}{\delta \varphi^i} + \frac{\delta \Gamma}{\delta K^a_\eta} \frac{\delta \Gamma}{\delta \eta^a} + \tfrac{\theta}{g} w^a \frac{\delta \Gamma}{\delta \overline{\eta}^a} \right] = 0$$

Hint. Use the functional chain rule and the definitions to show that: $\frac{\delta \Gamma}{\delta K} = \frac{\delta W}{\delta K}$*, and then use this relation explicitly.*