Exercise 1. ***Yukawa Functional determinant***

In order to understand the meaning and to compute the functional determinant for the Yukawa theory, and in general for any interacting field theory, it is more convenient to rewrite the theory as in the presence of a background field. In fact, this way we will be able to relate the functional determinant to a certain infinite set of Feynman diagrams.

To this aim, let us write the generating functional in the following equivalent form

$$Z[J,\eta,\bar{\eta}] \;=\; N \int D\phi\, e^{iS_0[\phi]+i\int d^4y\, J(y)\,\phi(y)}\, e^{-ig\int d^4x\,\phi(x)\,\frac{i\delta}{\delta\eta_x}\frac{\delta}{i\delta\bar{\eta}_x}}\, Z_0^F[\eta,\bar{\eta}] \tag{1}$$

where $S_0[\phi]$ is the classical action for the real Klein-Gordon field, and

$$Z_0^F[\eta,\bar{\eta}] = e^{-\int dx\int dy\;\bar{\eta}(x)\,\Delta_F(x-y)\,\eta(y)} \tag{2}$$

Therefore, we can now concentrate only on the last part of (1) and write

$$Z_\phi^F[\eta,\bar{\eta}] \;=\; N_\phi \int D\psi \int D\bar{\psi}\, e^{i\int dx\,\bar{\psi}(x)\left[i\partial\!\!\!/-m-g\phi(x)\right]\psi(x)+i\int dx\left[\bar{\eta}(x)\,\psi(x)+\bar{\psi}(x)\,\eta(x)\right]} \tag{3}$$

where ϕ plays now the role of a background field function. Consider in particular the case with no sources

$$\begin{aligned} Z_\phi^F[0,0] \;&=\; N_\phi \int D\psi \int D\bar{\psi}\, e^{i\int dx\,\bar{\psi}(x)\left[i\partial\!\!\!/-m-g\phi(x)\right]\psi(x)} \\ &\overset{\text{def}}{=}\; N_\phi\; \det\|i\partial\!\!\!/-m-g\phi\| \end{aligned} \tag{4}$$

where the definition in terms of the functional determinant has to be understood up to the Wick rotation to the Euclidean space, and the normalisation constant N_ϕ is usually fixed requiring that in the limit $\phi\to 0$, i.e. in the absence of the background field, we recover $Z_0^F[0,0]=1$. Now, in order to evaluate the determinant, consider the following symbolic equality

$$\begin{aligned} \frac{\det\|i\partial\!\!\!/-m-g\phi\|}{\det\|i\partial\!\!\!/-m\|} \;&=\; \det\|\mathbb{1}-g\,(i\partial\!\!\!/-m)^{-1}\phi\| \\ &=\; e^{\mathrm{Tr}\,\ln\|\mathbb{1}-g\,(i\partial\!\!\!/-m)^{-1}\phi\|} \\ &=\; e^{(-1)\sum_{n=1}^{\infty}\frac{1}{n}\,\mathrm{Tr}\left[g\,(i\partial\!\!\!/-m)^{-1}\phi\right]^n} \end{aligned} \tag{5}$$

Here Tr indicates both the the sum over spinor indices and the integration over space-time coordinates. If we now denote every term in the expansion as

$$A_\phi^n = (-1)\mathrm{Tr}\;\left[g\,(i\partial\!\!\!/-m)^{-1}\phi\right]^n \tag{6}$$

then the above equality can be written

$$\frac{\ln\det\|i\partial\!\!\!/-m-g\phi\|}{\ln\det\|i\partial\!\!\!/-m\|} = \sum_{n=1}^{\infty}\frac{A_\phi^n}{n} \tag{7}$$

Let's evaluate the first term of the expansion explicitly, so as to understand what it encodes.

$$\begin{aligned}
A^1_\phi &= (-1)\mathrm{Tr}\,\left[g\,(i\partial\!\!\!/ - m)^{-1}\phi\right] \\
&= -g\int dx\,\langle x|\mathrm{tr}\,(i\partial\!\!\!/ - m)^{-1}\phi\,|x\rangle \\
&= -g\int dx\int dy\,\langle x|\mathrm{tr}\,(i\partial\!\!\!/ - m)^{-1}\,|y\rangle\langle y|\phi|x\rangle \\
&= ig\int dx\int dy\,\phi(x)\,\delta(x-y)\,\mathrm{tr}\,\Delta_F(x-y) \\
&= ig\int dx\,\phi(x)\,\mathrm{tr}\,\Delta_F(0) \\
&\stackrel{\text{def}}{=} ig\,\mathrm{Tr}\,(\phi\Delta_F(0))
\end{aligned} \tag{8}$$

where now *tr* indicates the sum over spinor indices.

(a) Evaluate the second term of the expansion A^2_ϕ

(b) Show that it takes the form

$$A^2_\phi = (-i\,g)^2\int\frac{dk}{(2\pi)^4}\,\tilde\phi(k)\,\tilde\phi(-k)\,(-1)\int\frac{dp}{(2\pi)^4}\;\mathrm{tr}\;\left[\frac{i}{p\!\!\!/ - m + i\varepsilon}\,\frac{i}{p\!\!\!/ + k\!\!\!/ - m + i\varepsilon}\right] \tag{9}$$

Which kind of Feynman diagrams does this represent?

Hint. Consider the following change of variables

$$\bar x = \frac{x_1 + x_2}{2} \qquad x = x_1 - x_2$$

where x_1, x_2 are the coordinate integration variables.

We can then iterate and find

$$\begin{aligned}
A^n_\phi &= (-i\,g)^n\int\frac{dk_1}{(2\pi)^4}\,\tilde\phi(k_1)\ldots\int\frac{dk_{n-1}}{(2\pi)^4}\,\tilde\phi(k_{n-1})\,\tilde\phi(k_1+\cdots+k_{n-1}) \\
&\times\;(-1)\int\frac{dp}{(2\pi)^4}\;\mathrm{tr}\;\left[\tilde\Delta_F(p)\,\tilde\Delta_F(p+k_1)\;\ldots\;\tilde\Delta_F(p+k_1+\cdots+k_{n-1})\right] \\
&= (-1)(-i\,g)^n\,\mathrm{Tr}(\phi\Delta_F)^n
\end{aligned} \tag{10}$$

This way, we can eventually write our initial symbolic equality as

$$\frac{\det\,\|i\partial\!\!\!/ - m - g\phi\|}{\det\,\|i\partial\!\!\!/ - m\|} = e^{(-1)\sum_{n=1}^{\infty}\frac{(-ig)^n}{n}\;\mathrm{Tr}\;(\phi\Delta_F)^n} \tag{11}$$

This is a perturbative expansion in the Yukawa coupling.

(c) Which kind of Feynman rules can we learn from the above expansion?

Exercise 2. ***Dual Field Strength Tensors***

(a) In abelian gauge theory consider the dual tensor $\tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}$. Show that

$$F^{\mu\nu}\tilde{F}_{\mu\nu} = \partial_\mu K^\mu \tag{12}$$

with $K^\mu = \epsilon^{\mu\nu\rho\sigma}A_\nu F_{\rho\sigma}$.

Hint. Use that contractions of the ϵ tensor with a symmetric tensor vanish.

(b) In a non-abelian gauge theory consider the dual tensor $\tilde{G}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}G_{\rho\sigma}$. Show that

$$Tr\left(G^{\mu\nu}\tilde{G}_{\mu\nu}\right) = \partial_\mu K^\mu \tag{13}$$

with $K^\mu = \epsilon^{\mu\nu\rho\sigma}Tr\left[G_{\nu\rho}A_\sigma + \frac{2}{3}igA_\nu A_\rho A_\sigma\right]$.

Hint. Use the cyclicity of the trace.

Remark: The dual field tensor in QED corresponds to the interchange of E and B; one often says that E and B are dual.
The term $F^{\mu\nu}\tilde{F}_{\mu\nu}$ is called the θ-term, $Tr\left(G^{\mu\nu}\tilde{G}_{\mu\nu}\right)$ goes by the name of $\bar{\theta}$-term. They are manifestly Lorentz invariant objects. In QED, it corresponds to $E \cdot B$. These terms are in fact CP-violating and therefore, their coupling constant must be very small (due to experimental constraints on CP-violation). The apparent lack of a reason for the $\bar{\theta}$-term to be so small is called *the strong CP problem.*
The θ-term also arises in the context of anomalies where they correspond to the right hand side of the anomaly equation (see e.g. Peskin & Schroder, (19.45)). Anomalies are symmetry violations that only arise at the one-loop level. They are very important in quantum field theory. For instance, the masses of the nucleons are due to an anomaly in the energy-momentum tensor.

Exercise 3. ***Gauge Invariance of the measure in Yang-Mills theory***

Consider a gauge transformation $A^a_\mu \to A^{a\prime}_\mu$. Prove that

$$\mathcal{D}A^a_\mu = \mathcal{D}A^{a\prime}_\mu \tag{14}$$

You only need to consider an infinitesimal gauge transformation.

Exercise 4. ***Yang-Mills equation of motion***

Given that the action of Yang-Mills theory has the form:

$$S_{YM} = \frac{1}{2}\int d^4x \ \mathrm{Tr}\left[G_{\mu\nu}G^{\mu\nu}\right]$$

determine the equations of motion from the extremality condition: $\delta S_{YM} = 0$.

Hint. Use the anti-symmetry of f^{abc} and $G_{\mu\nu}$.