Exercise 1. ***Renormalization of ϕ^4 theory***

Let's consider the $\lambda\phi^4$ theory:

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4 \tag{1}$$

Write up to two loop order all the (connected) diagrams contributing to the 2, 3 (if any) and 4-point function and compute the superficial degree of divergence for each of them. Are the counterterms for the mass and coupling constant sufficient to cancel all the divergences at two loop order?

Exercise 2. ***Renormalization in QCD at one loop: The Propagators***

In this exercise series we will learn how to compute the UV divergent diagrams at one loop in QCD and how to renormalize these. For this we will absorb these divergencies into the physical quantities and write the bare Lagrangian parameters in terms of renomalized ones such as masses and coupling constants. The divergent diagrams include the corrections to the propagators and the corrections to the vertices. In this exercise we will look at the propagators only.

a) The Quark Propagator.

Consider the one loop correction to the quark propagator as shown in Figure 2. We can separate the self energy as a combination of the vector and a scalar pieces:

$$\Sigma(p, m_0) = \Sigma_V(p^2)\not{p} + \Sigma_S(p^2)m_0 \tag{2}$$

where p is the momentum of the external fermion and m_0 is the bare mass for the fermion.

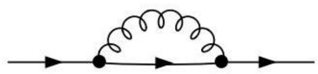

Figure 1: One loop quark self energy

i) Write down the expression for $\Sigma(p, m_0)$ by applying the Feynman rules for QCD in Feynman gauge, i.e. $\xi = 1$. Convince yourself that this diagram is indeed divergent.

ii) Show that in $d = 4 - 2\epsilon$ dimensions:

$$\Sigma_S = \frac{\alpha_s}{4\pi}\mu^{2\epsilon}\, C_F\,(4\pi)^\epsilon\,\Gamma(\epsilon)\,(4-2\epsilon)\int_0^1 dx(m_0^2 x - p^2 x(1-x))^{-\epsilon} \tag{3}$$

$$\Sigma_V = \frac{\alpha_s}{4\pi}\mu^{2\epsilon}\, C_F\,(4\pi)^\epsilon\,\Gamma(\epsilon)\,(2\epsilon-2)\int_0^1 dx(m_0^2 x - p^2 x(1-x))^{-\epsilon}(1-x) \tag{4}$$

For this, you will need to recall the identities of the d-dimensional γ matrices. Apply Feynman parametrization to the expression you found for $\Sigma(p, m_0)$ in part $i)$ and then bring it to the form in Equation 2.

iii) Expand your results for Σ_S and Σ_V in ϵ. Observe that there are $1/\epsilon$ terms, which represent the divergences in the physical limit, $\epsilon \to 0$.

iv) The UV divergences are renormalized by absorbing them into the physical quantities. We define:

$$\psi_0 = \sqrt{Z_2}\psi \tag{5}$$
$$m_0 = Z_m m \tag{6}$$

where ψ and m are wave function and mass for the quark respectively. With this definition show that the quark propagator becomes:

$$S_q(p) = \frac{i}{Z_2\not{p} - Z_2 Z_m m - \Sigma(p,m)} \tag{7}$$

Hint: Remember that using the geometic series we can express the full propagator in terms of sum of 1 particle irreducible diagrams:

$$S_q(p) = \frac{i}{\not{p} - m_0 - \Sigma(p,m_0)} \tag{8}$$

where $\Sigma(p,m_0)$ is the sum of all 1PI diagrams. Up to second order in perturbation theory $\Sigma(p,m_0)$ actually becomes the self energy diagram that is shown in the Figure 2.

v) From the results you found in part iv), show that:

$$Z_2 = 1 - \frac{\alpha_s}{4\pi} C_F \left(\frac{1}{\epsilon} + \text{finite}\right) \tag{9}$$
$$Z_m = 1 - \frac{\alpha_s}{4\pi} 3C_F \left(\frac{1}{\epsilon} + \text{finite}\right) \tag{10}$$

so that the divergencies are cancelled.

b) The Gluon Propagator.

There are three self energy diagrams for gluon at one loop as shown in Figure 2.

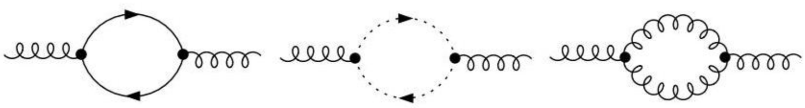

Figure 2: One loop gluon self energy

i) Let $\Pi^{\mu\nu}$ express the one loop gluon self energy. Write down an expression for each diagram contributing to $\Pi^{\mu\nu}$, i.e. $\Pi_1^{\mu\nu}$, $\Pi_2^{\mu\nu}$ and $\Pi_3^{\mu\nu}$ by applying Feynman rules.

ii) Compute the gluonic contribution, i.e. $\Pi_3^{\mu\nu}$ in d-dimensions.

∗ First work out the numerator and write $\Pi_3^{\mu\nu}$ as:

$$\Pi_3^{\mu\nu} = \frac{g_0^2}{2} N_c \delta_{cd} \int \frac{d^d k}{(2\pi)^d} \frac{A^{\mu\nu}}{k^2 (p+k)^2} \tag{11}$$

where k and p are the loop and external momentas respectively.

Keeping the tensor structure apply Feynman parametrization and make the necessary shifting of the loop momentum. Note that the terms linear in the loop momentum are odd and therefore vanishes:

$$\int \frac{d^d k}{(2\pi)^d} \frac{k^\mu}{(k^2 - \Delta)^2} = 0 \tag{12}$$

This means you can make the replacement:

$$k^\mu k^\nu = \frac{k^2}{d} g^{\mu\nu} + \text{terms linear in k which vanish} \tag{13}$$

You should find:

$$\Pi_3^{\mu\nu} = \frac{g_0^2}{2} N_c \delta_{cd} \int dx \int \frac{d^d k}{(2\pi)^d} \frac{g^{\mu\nu}\left[\left(2 + \frac{4d-6}{d}\right) k^2 + 2\Delta + 5p^2\right] + M^{\mu\nu}}{(k^2 - \Delta^2)^2} \tag{14}$$

where

$$M^{\mu\nu} = \frac{p^\mu p^\nu}{p^2}\left(\Delta(4d-6) + p^2(d-6)\right), \quad \Delta = -p^2 x(1-x) \tag{15}$$

∗ Now make the Wick rotation to perform the loop integration in the Euclidean space. Note that:

$$\int \frac{d^d k}{(2\pi)^d} \frac{k^2}{(k^2 + \Delta)^2} = \frac{(2-\epsilon)\Delta}{\epsilon - 1} \Delta^{-\epsilon} (4\pi)^\epsilon \Gamma(\epsilon) \tag{16}$$

$$\int \frac{d^d k}{(2\pi)^d} \frac{1}{(k^2 + \Delta)^2} = \Delta^{-\epsilon} (4\pi)^\epsilon \Gamma(\epsilon) \tag{17}$$

∗ Finally you need to perform the integration over the Feynman parameter x. Your result should be

$$\Pi_3^{\mu\nu} = \frac{i g_0^2}{2} N_C \delta_{cd}\, p^2 \left[g^{\mu\nu}(19 - 12\epsilon) + \frac{p^\mu p^\nu}{p^2}(-22 + 14\epsilon)\right] \tag{18}$$

$$\Gamma(\epsilon)\,(4\pi)^\epsilon\,(-p^2)^{-\epsilon}\,\frac{B(2-\epsilon, 2-\epsilon)}{1-\epsilon} \tag{19}$$

where $B(2-\epsilon, 2-\epsilon)$ is the Beta function:

$$B(2-\epsilon, 2-\epsilon) = \int_0^1 dx\, x^{1-\epsilon}\,(1-x)^{1-\epsilon} \tag{20}$$

iii) Now compute the ghost contribution $\Pi_2^{\mu\nu}$ following the same steps before. (Do not forget the minus sign for the fermion and the ghost loop!) You should find:

$$\Pi_2^{\mu\nu} = \frac{1}{2}\frac{g_0^2 N_C\, \delta_{cd}}{16\pi^2}(4\pi)^\epsilon (-p^2)^{-\epsilon}\,\Gamma(\epsilon)\,\left[g^{\mu\nu} p^2 + 2(1-\epsilon) p^\mu p^\nu\right]\,\frac{B(2-\epsilon, 2-\epsilon)}{1-\epsilon} \tag{21}$$

iv) Show that the sum of $\Pi_2^{\mu\nu}$ and $\Pi_3^{\mu\nu}$ can be brought to the form:

$$\Pi^{\mu\nu}(p) = \left(p^2 g_{\mu\nu} - p_\mu p_\nu\right)\Pi(p^2) \tag{22}$$

where p is the momentum of the external gluon. This implies that:

$$\Pi_{\mu\nu}p^\mu = 0. \tag{23}$$

which is a consequence of the Ward identity in QCD and tells that the gluons are transverse. It is important to notice that individually the second and third diagrams cannot be brought to this form. This means to express a physical gluon propagator we need the ghost contribution!

v) Now compute the fermionic contribution $\Pi_1^{\mu\nu}$. Take the fermions running in the loop massless so that you can write:

$$\Pi_1^{\mu\nu} = -\frac{g_0^2}{4\pi^2}\frac{\delta_{ab}}{2} n_f\,\Gamma(\epsilon)\,(4\pi)^\epsilon\,\left[g^{\mu\nu}p^2 - p^\mu p^\nu\right]\,(-p^2)^{-\epsilon}\,B(2-\epsilon, 2-\epsilon) \tag{24}$$

vi) We define the renomalized wave function for the gluon field as:

$$A_0 = Z_A^{1/2}A \tag{25}$$

which leads to renormalised propagator at leading order:

$$D_{ab}^{\mu\nu} = \frac{\delta_{ab}(-ig^{\mu\nu})}{Z_A p^2} \tag{26}$$

Then up to second order in perturbation theory the gluon propagator is:

$$S_{ab}^{\mu\nu}(p) = \frac{\delta_{ab}(-ig^{\mu\nu})}{Z_A p^2} + \frac{\delta_{ac}(-ig^{\mu\sigma})}{Z_A p^2}\,\Pi_{\sigma\tau}^{cd}(p)\,\frac{\delta_{db}(-ig^{\tau\nu})}{Z_A p^2} \tag{27}$$

Using the transversality:

$$\Pi_{\sigma\tau}^{cd}(p) = \delta^{cd}\left(-g_{\sigma\tau} + \frac{p_\sigma p_\tau}{p^2}\right)\Pi(p^2) \tag{28}$$

Now using this and expressing $S_{ab}^{\mu\nu}(p)$ as a geometric sum we have:

$$S_{ab}^{\mu\nu}(p) = \frac{\delta_{ab}(-ig^{\mu\nu})}{Z_A p^2 + \Pi(p^2)} \tag{29}$$

Now combine your results for $\Pi(p^2)$ from part iv) to determine Z_A so that the divergences cancel:

$$Z_A = 1 - \frac{\alpha_s}{4\pi}\frac{1}{\epsilon}\left(\frac{2}{3}n_f - \frac{5}{3}N_C\right) + \text{finite} \tag{30}$$