# Third Exercise Sheet due to 13. March

**Exercise 1 (Detection of a binary coherent signal, part1)** In the lecture we discussed description of a measurement based on the notion of POVM. It is the aim of this and the following exercise to show that concrete realization of such measurement can be difficult. In fact only very few "optimal" detection schemes are known. We are going to describe a device called Dolinar receiver that distinguishes two modes of coherent light. As a preliminary part we describe what is called the binary hypothesis testing.

Let $|\psi_0\rangle$, $|\psi_1\rangle$ be two input states that we aim to distinguish by the following procedure. We pick a POVM measurement consisting of two operators $\Pi_0 + \Pi_1 = \mathsf{Id}$. Given measurement result 0 we guess state $|\psi_0\rangle$ and given measurement result 1 we guess state $|\psi_1\rangle$. Two (among many) quantities of interest are the *false alarm probability* equal to the probability of outcome 1 given that the state was 0, and the *detection probability* equal to the probability of outcome 1 given that the state was 1. By rules of quantum mechanics,

$$\begin{aligned} Q_0 &:= false\,alarm\,prob. = \langle\psi_0|\Pi_1|\psi_0\rangle, \\ Q_d &:= detection\,prob. = \langle\psi_1|\Pi_1|\psi_1\rangle. \end{aligned}$$

We will study the average probability of error given by

$$P_e = \zeta_0 Q_0 + \zeta_1(1 - Q_d),$$

where $\zeta_0, \zeta_1$ are prior probabilities of states $|\psi_0\rangle$ and $|\psi_1\rangle$ respectively. Assume unbiased prior, i.e. $\zeta_0 = \zeta_1 = 1/2$. Find a measurement scheme (i.e. POVM) that minimizes $P_e$ and show that the minimum is

$$P_e = \frac{1}{2}(1 - \sqrt{1 - |\langle\psi_0|\psi_1\rangle|^2}).$$

You can also try the case of general $\zeta_1$, $\zeta_2$ but expect little bit messy expressions. The final result for that case is

$$P_e = \frac{1}{2}(1 - \sqrt{1 - 4\zeta_0\zeta_1|\langle\psi_0|\psi_1\rangle|^2}).$$

In the follow up exercise we will study a device that implements this optimal detection procedure for a binary coherent light signal.