First Exercise Sheet due to 27. February

Exercise 1 (Shallow pocket model) *Consider a system* $\bar{S} = S \vee E$ *with a Hamiltonian*

$$H = A \otimes B,$$

where $A = \sum_i a_i |\phi_i\rangle\langle\phi_i|$ *is an operator with a discrete, non-degenerate (*$a_i \neq a_j$ *for* $i \neq j$*) spectrum. Let* $|\tilde{\psi}_t\rangle = \exp(-iHt)(|\psi_S\rangle \otimes |\psi_E\rangle)$ *be a solution of the Schrödinger equation with an initial condition* $|\psi_S\rangle \otimes |\psi_E\rangle \in \mathcal{H}_S \otimes \mathcal{H}_E$*.*

1. *Compute* $\rho_t := \mathrm{Tr}_E(|\tilde{\psi}_t\rangle\langle\tilde{\psi}_t|)$*.*

2. *Let* $\rho_{ij}(t) := \langle\phi_i|\rho_t|\phi_j\rangle$ *be elements of the density matrix in the eigenbasis of A. Show that* $\rho_{ii}(t)$ *is constant and find conditions*[1] *under which* $\rho_{ij}(t) \to 0$ *as* $t \to \infty$ *for off diagonal terms (*$i \neq j$*). Such process is called dephasing and we will encounter it repeatedly.*

   In particular, show that if $\mathcal{H}_E = L^2(-\infty, \infty)$ *is a Hilbert space of a particle on a real line,* $B = X$ *is a position operator and*

   $$\langle\psi_E|x\rangle = \sqrt{\frac{\gamma}{\pi}} \frac{1}{x + i\gamma},$$

   then $\rho_{ij}(t)$ *decays exponentially.*

3. *Convince yourself that the method used to compute* ρ_t *and the conclusions about the decay of the off diagonal components are the same also for a more general Hamiltonian*

   $$H = H_S \otimes \mathsf{Id} + A \otimes B + \mathsf{Id} \otimes H_E$$

   provided $[H_S, A] = 0$ *and* $[H_E, B] = 0$*.*

[1] If you are mathematically minded you will want to express these conditions in terms of the spectral measure of the operator B, if physically minded then for example in scattering terms.

About the Zero Excercise

During the zero exercise on 20.February we shall discuss properties of the tensor product, the partial trace and positive operators (states). Below are some sample questions. If those are all clear there is no need in coming.

- Compute the partial trace of a Bell state $|\alpha\rangle\langle\alpha|$ where

$$|\alpha\rangle = \frac{1}{\sqrt{2}}(|0\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle);$$

- $\mathrm{Tr}(A \otimes B) = \mathrm{Tr}(A)\mathrm{Tr}(B)$;
- $Dim(\mathcal{H}_S \otimes \mathcal{H}_E) = Dim(\mathcal{H}_S)Dim(\mathcal{H}_E)$;
- A positive operator is always Hermitian ($A \geq 0 \implies A = A^*$);
- A two by two matrix is positive if and only if it has a form $n_0\mathsf{Id} + \vec{n} \cdot \vec{\sigma}$ for some real number n_0 and a real vector $\vec{n}$ such that $n_0 \geq |\vec{n}|$. Above, $\vec{\sigma}$ stands for a vector of Pauli matrices.
- Pauli matrices satisfy a relation

$$(\vec{n} \cdot \sigma)(\vec{m} \cdot \sigma) = i(\vec{n} \times \vec{m}) \cdot \vec{\sigma} + (\vec{n} \cdot \vec{m})\mathsf{Id}$$