Deformation of an elastic medium when point force is applied in the bulk

Equilibrium equation is

$$\nabla^2 \vec{u} + \frac{1}{1-2\sigma}\,\mathrm{grad}\,\mathrm{div}\,\vec{u} = -\frac{2(1+\sigma)}{E}\vec{F}\delta(r)$$

point force

We should find Green's function.
It is convenient to do it in Fourier space.

$$k^2 \vec{u} + \frac{1}{(1-2\sigma)}\vec{k}(\vec{k}\cdot\vec{u}) = \frac{2(1+\sigma)}{E}\vec{F}$$

To solve it we extract $\vec{k}\cdot\vec{u}$

$$k^2(\vec{k}\cdot\vec{u}) + \frac{k^2(\vec{k}\cdot\vec{u})}{1-2\sigma} = \frac{2(1+\sigma)}{E}(\vec{k}\cdot\vec{F})$$

Then

$$(\vec{k}\cdot\vec{u}) = \frac{(1+\sigma)(1-2\sigma)}{E(1-\sigma)} \frac{\vec{k}\cdot\vec{F}}{k^2}$$

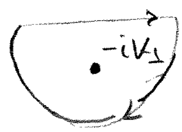
Substituting it back in the equation for u we obtain

$$u(k) = \frac{2(1+\sigma)}{E}\left[\frac{\vec{F}}{k^2} - \frac{1}{2(1-\sigma)}\frac{\vec{k}(\vec{k}\cdot\vec{F})}{k^4}\right]$$

Calculating Fourier transform back

$$\int \frac{e^{-i\vec{k}\vec{r}}}{k^2}\frac{d^3k}{(2\pi)^3} = \frac{1}{(2\pi)^3}\int \frac{e^{-ik_z r}\, dk_z\, d^2k_\perp}{k_z^2+k_\perp^2} =$$

$$= \frac{2\pi}{(2\pi)^3}\int \frac{e^{-k_\perp r}}{2k_\perp}\, d^2k_\perp$$



$$= \frac{2\pi}{(2\pi)^3}\int_0^\infty e^{-k_\perp r}\,\frac{\pi k_\perp\, dk_\perp}{k_\perp} =$$

$$= \frac{2\pi^2}{(2\pi)^3}\,\frac{1}{r} = \frac{1}{4\pi r}$$ (Also known from electrostatic)

Because Fourier transform of ∇f is $i\vec{k}\, f(k)$, Fourier transform of $\vec{k}\,(\vec{k}\cdot\vec{F})\, f(k)$ is $-\nabla(\vec{F}\cdot\vec{\nabla})\, f(r)$

Then

$$\int e^{-i\vec{k}\cdot\vec{r}}\,\frac{\vec{k}\cdot(\vec{k}\cdot\vec{F})}{}\,\frac{1}{k^4}\,\frac{d^3k}{(2\pi)^3} = -\vec{\nabla}(\vec{F}\cdot\vec{\nabla})\int \frac{e^{-i(\vec{k}\cdot\vec{r})}}{k^4}\,\frac{d^3k}{(2\pi)^3}$$

The last integral is

$$\frac{1}{(2\pi)^3}\int \frac{e^{-ik_z r}\,dk_z\, d^2k_\perp}{(k_z^2+k_\perp^2)^2} = \frac{1}{(2\pi)^3}\int \frac{e^{-ik_z r}\,dk_z\ \pi\, d(k_\perp^2)}{(k_z^2+k_\perp^2)^2} =$$

$$= \frac{\pi}{(2\pi)^3}\underbrace{\int \frac{e^{-ik_z r}\,dk_z}{k_z^2}}_{I}$$

$$\frac{\partial^2 I}{\partial r^2} = -\int e^{-ik_z r}\,dk_z = -2\pi\,\delta(r) \Rightarrow$$

$$I = -\pi\,|r|$$

Combining all factors we obtain

$$\int e^{-i\vec{k}\cdot\vec{r}} \frac{\vec{k}\cdot(\vec{k}\cdot\vec{F})}{k^4} \frac{d^3k}{(2\pi)^3} = \frac{\pi^2}{(2\pi)^3} \vec{\nabla}\cdot(\vec{F}\cdot\vec{\nabla})\, r =$$

$$= \frac{1}{8\pi} \frac{\vec{F} - \vec{n}(\vec{n}\cdot\vec{F})}{r}, \text{ with } \vec{n} = \frac{\vec{r}}{r}$$

- Substituting back we arrive at

$$u = \frac{(1+\sigma)}{2\pi E}\left[\frac{\vec{F}}{r} - \frac{1}{4(1-\sigma)} \frac{\vec{F} - (\vec{n}\vec{F})\vec{n}}{r}\right] =$$

$$\vec{u} = \frac{1+\sigma}{8\pi E(1-\sigma)} \frac{(3-4\sigma)\vec{F} + \vec{n}(\vec{n}\cdot\vec{F})}{r}$$

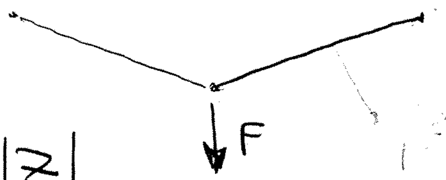
- Note that displacement decays only as $\frac{1}{r}$. This is the property of Laplace equation in 3d

In 1 dimension (string)

$$F = \int \left[\frac{C}{2} \left(\frac{du}{dz} \right)^2 + F\delta(z) u \right] dz$$

Then equilibrium equation is

$$C \frac{d^2 u}{dz^2} = F\delta(z)$$



- It has solution $u = -\alpha |z|$

Integrating around $z = 0$ we obtain

$$C \left. \frac{du}{dz} \right|_{-0}^{+0} = F \Rightarrow \alpha = \frac{F}{2C}$$

- In this case displacement from the action of the point force grows linearly with the distance.