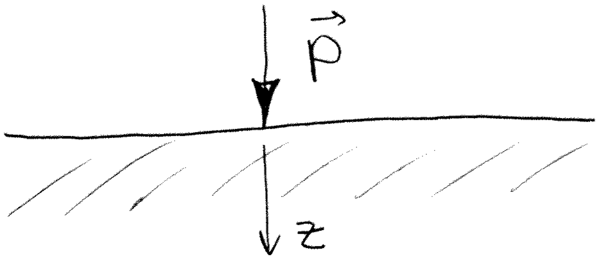
Point force applied to the surface

J. Boussinesq (1879)



We should solve equation

$$\mu \nabla^2 \vec{u} + (\mu+\lambda)\,\mathrm{grad}\,\mathrm{div}\,\vec{u} = 0$$

with the boundary conditions

$\sigma_{rz} = \sigma_{\varphi z} = 0$ for $z=0$ $\quad \sigma_{zz}(z=0) = -P\delta^2(r)$

It is convenient to use cylindrical coordinates (r, φ, z)

Taking $\mathrm{div}\left[\mu \nabla^2 u + (\mu+\lambda)\,\mathrm{grad}\,\mathrm{div}\,u\right] = 0$

we obtain

$$(2\mu+\lambda)\nabla^2 \mathrm{div}\,u = 0 \quad \Rightarrow$$

$$\underline{\nabla^2 \mathrm{div}\,u = 0}$$

From the other side using z component of the equilibrium equation we have

$$\nabla^2 u_z = -\frac{\mu+\lambda}{\mu}\frac{\partial}{\partial z}(\mathrm{div}\,\vec{u})$$

Let's look for the solution of the Laplace equation on $\mathrm{div}\,\vec{u}$ in the form

$$\mathrm{div}\, u = -\alpha \frac{\partial}{\partial z}\frac{1}{R} = \alpha \frac{z}{R^3}$$

where $R = \sqrt{r^2+z^2}$ and constant α to be determined from the boundary conditions.

With this Ansatz

$$\nabla^2 u_z = \alpha\frac{(\mu+\lambda)}{\lambda}\frac{\partial^2}{\partial z^2}\left(\frac{1}{R}\right)$$

Since $\nabla^2 \frac{R}{2} = \frac{1}{R}$ we can eliminate ∇^2 and obtain

$$u_z = \alpha\frac{(\mu+\lambda)}{2\lambda}\frac{\partial^2 R}{\partial z^2} = \alpha\frac{(\mu+\lambda)}{2\mu}\left[\frac{1}{R} - \frac{z^2}{R^3}\right]$$

We can add any harmonic function to it, so we choose

$$u_z = \frac{\gamma}{R} - \alpha\frac{(\mu+\lambda)}{2\mu}\frac{z^2}{R^3}$$

with γ to be determined later

With known u_z and $\operatorname{div}\vec{u}$ we can find the radial component u_r.

Indeed

$$\operatorname{div}\vec{u} = \frac{1}{r}\frac{\partial(r u_r)}{\partial r} + \frac{\partial u_z}{\partial z}$$

$$\frac{\partial u_z}{\partial z} = -\left(\gamma + \alpha\frac{(\mu+\lambda)}{\mu}\right)\frac{z}{R^3} + \alpha\frac{(\mu+\lambda)}{\mu}\frac{3}{2}\frac{z^3}{R^5}$$

$$\operatorname{div}\vec{u} = \alpha\frac{z}{R^3}$$

thus

$$\frac{\partial}{\partial r}(r u_r) = r\left(\operatorname{div}\vec{u} - \frac{\partial u_z}{\partial z}\right) = -\left[\gamma + \alpha\frac{(2\mu+\lambda)}{\mu}\right] z\frac{\partial}{\partial r}\frac{1}{R} +$$

$$+ \alpha\frac{\mu+\lambda}{2\mu} z^3 \frac{\partial}{\partial r}\frac{1}{R^3}$$

As a result

$$r u_r = -\left[\gamma + \alpha\frac{(2\mu+\lambda)}{\mu}\right]\frac{z}{R} + \alpha\frac{(\mu+\lambda)}{2\mu}\frac{z^3}{R^3} + \varphi(z)$$

where $\varphi(z)$ is integration constant

Since for $r=0$ $r u_r = 0$

$$\varphi(z) = \gamma + \alpha\frac{(2\mu+\lambda)}{\mu} - \alpha\frac{\mu+\lambda}{2\mu} = \gamma + \alpha\frac{(3\mu+\lambda)}{2\mu}$$

Thus

$$u_r = \left(\gamma + \alpha\frac{[3\mu+\lambda]}{2\mu}\right)\frac{1}{r} - \left[\gamma + \alpha\frac{(2\mu+\lambda)}{\mu}\right]\frac{z}{rR} + \frac{\alpha(\mu+\lambda)}{2\mu}\frac{z^3}{rR^3}$$

One can check, that for $r \to 0$ $u_r \propto r$

Now we should satisfy the boundary conditions

Since $\sigma_{rz} = 2\mu u_{rz} = \mu\left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right) = 0$ at $z=0$

we obtain

$$-\left(\gamma + \alpha\frac{(2\mu+\lambda)}{\mu}\right)\frac{1}{r^2} - \frac{\gamma}{r^2} = 0$$

Taking derivatives we used the fact that $R \approx r\left(1+\frac{z^2}{2r}\right) \Rightarrow$ for $z \to 0$ $R \to r$ and we should take derivatives only of z but not of R.

As a result

$$\gamma = -\alpha \frac{(2\mu+\lambda)}{2\mu}$$

Then the expressions for u_z and u_r are simplified

$$u_z = -\frac{\alpha}{2R}\left(\frac{2\mu+\lambda}{\mu} + \frac{(\mu+\lambda)}{\mu}\frac{z^2}{R^2}\right)$$

$$u_r = \frac{\alpha}{2r}\left(1 - \frac{(2\mu+\lambda)}{\mu}\frac{z}{R} + \frac{\mu+\lambda}{\mu}\frac{z^3}{R^3}\right)$$

$$\sigma_{zz} = 2\mu\frac{\partial u_z}{\partial z} + \lambda \operatorname{div} u = 3\alpha(\mu+\lambda)\frac{z^3}{R^5}$$

At the surface $\sigma_{zz}(z=0) = -P\,\delta^2(r)$

Really $\int d^2r \frac{z^3}{(z^2+r^2)^{5/2}} = \pi \int_1^\infty dt \frac{1}{t^{5/2}} = \frac{2}{3}\pi$

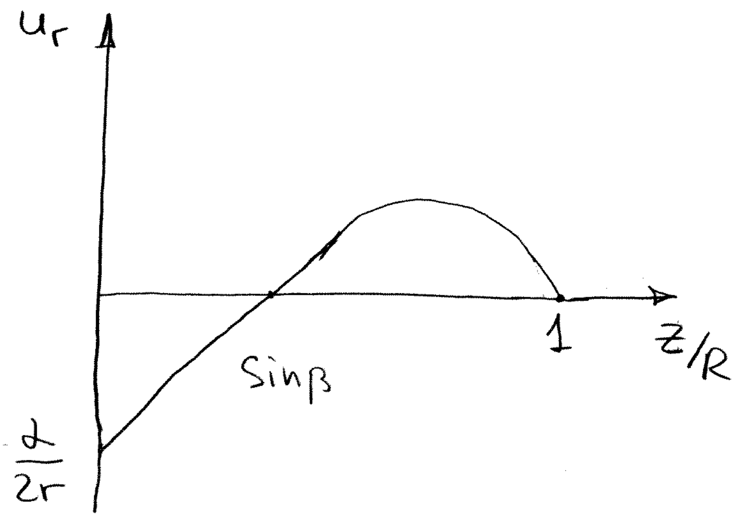
and $\frac{z^3}{R^5} = 0$ for $z=0$, $r \neq 0$

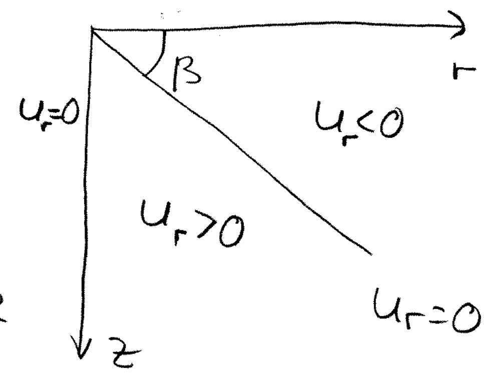
Then $\alpha = -\frac{P}{2\pi(\mu+\lambda)}$

Note, that for $R \to 0$, $u_z, u_r \sim \frac{\alpha}{R}$ and diverge. We can use linearized elasticity theory only if $u_{ij} \sim \frac{\alpha}{R^2} << 1 \Rightarrow$ R should be bigger then $\sqrt{\alpha}$. 50

For $z = 0$ $u_r = \frac{\alpha}{2r} < 0$

For $z > 0$ it depends on the angle





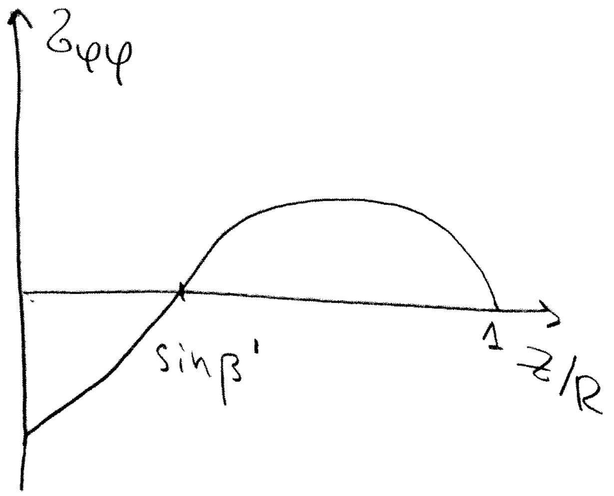
At the surface we have rings with radius r are shrinking ($u_r<0$) and deeper inside they expand. The neutral angle β is given by

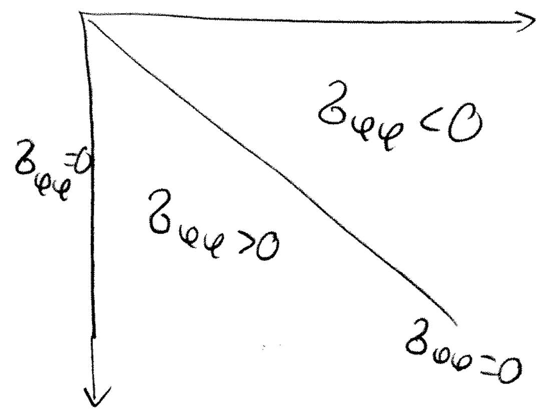
$$\frac{z}{R} = \sin\beta = \sqrt{\frac{1}{4} + \frac{\mu}{\mu+\lambda}} - \frac{1}{2}$$

Analogously

$$\sigma_{\varphi\varphi} = 2\mu \frac{u_r}{r} + \lambda \,\mathrm{div}\, u =$$

$$= \frac{2\mu}{r^2}\left(1 - 2\frac{z}{R} + \frac{z^3}{R^3}\right)$$

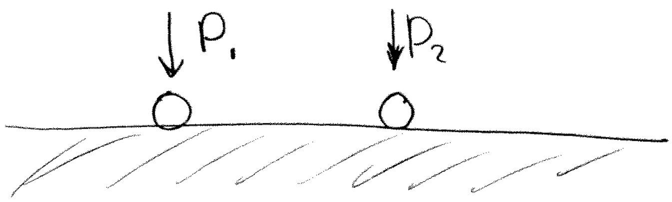




Here neutral angle is

$$\sin\beta' = \frac{\sqrt{5}-1}{2}, \quad \beta' \approx 38.2^\circ$$

Consider two balls at the surface



Then $P(r) = \vec{P}\left[\delta^2(r-r_1) + \delta^2(r-r_2)\right] = \vec{P}_1 + \vec{P}_2$

We will look for solution as superposition

$$\vec{u} = \vec{u}_1 + \vec{u}_2$$

Where $u_{1,2}$ are solution for the force applied at $r_{1,2}$. In the quadratic form

$$\lambda\,(\mathrm{div}\,\vec{u})^2 + \mu\left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i}\right)^2$$

interaction is due to the mixed term

$\partial u_1\, \partial u_2$, integrating by parts

$$F(u_1+u_2) = F(u_1) + F(u_2) +$$

$$\int \underbrace{\left[-\lambda \frac{\partial}{\partial x_j}\frac{\partial u_{1k}}{\partial x_k} - \mu \frac{\partial}{\partial x_i}\left(\frac{\partial u_{1j}}{\partial x_i} + \frac{\partial u_{1i}}{\partial x_j}\right)\right]}_{=0} u_{2j}\, dV$$

$=0$ (equilibrium equation in the volume

$$+ \int \underbrace{\left[\lambda\left(\frac{\partial u_{1k}}{\partial x_k}\right) n_j + \mu\left(\frac{\partial u_{1j}}{\partial x_i} + \frac{\partial u_{1i}}{\partial x_j}\right) n_i - P_j\,\delta^2(r-r_1)\right]}_{=0} u_{2j}\, dS - \int P_j\,\delta^2(r-r_2)\, u_{1j}\, dS$$

$=0$ boundary condition

As a result interaction is given by the last term

$$\Delta U_{int} = -\int P_j \delta^2(r-r_2) u_{i,j}\, ds = -P u_{i,z}(r_2) =$$

$$= -\frac{P^2}{4\pi} \frac{2\mu+\lambda}{\mu(\mu+\lambda)} \frac{1}{|r_1-r_2|}$$

It is attraction