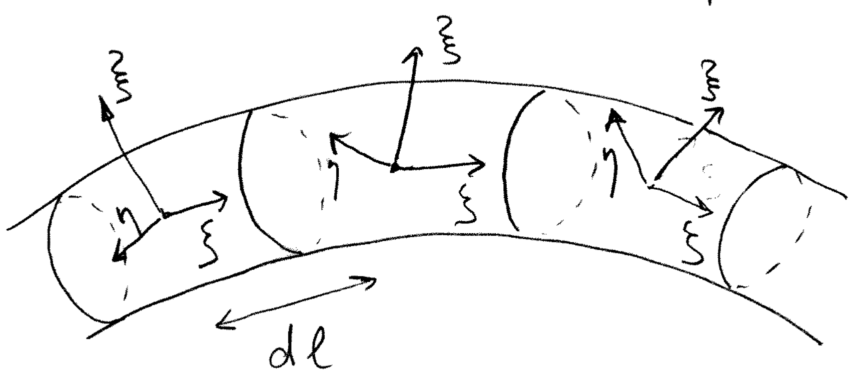
The energy of a deformed rod

General description

We divide the rod into infinitesimal elements bounded by two adjacent cross-sections. For each of those elements we use a cordinate system ξ, η, ζ. ζ axis is parallel to the axis of the rod. All the systems are parallel in the undeformed state



Let $d\vec{\varphi}$ be the vector of the angle of relative rotations of the two systems at a distance dl apart. Deformation is determined by the rate of rotation of the coordinate axes along the rod $\frac{d\vec{\varphi}}{dl}$

We can construct quadratic form out of $\frac{d\vec{\varphi}}{dl}$

$$F = \int \left[\frac{1}{2} I_1 E \left(\frac{d\varphi_\xi}{dl}\right)^2 + \frac{1}{2} I_2 E \left(\frac{d\varphi_\eta}{dl}\right)^2 + \frac{1}{2} C \left(\frac{d\varphi_\zeta}{dl}\right)^2 \right] dl$$

First two terms in the expression above can be written as $\int \left[\frac{1}{2} I_1 E \left(\frac{d\vec{\tau}_\eta}{dl}\right)^2 + \frac{I_2}{2} E \left(\frac{d\tau_\xi}{dl}\right)^2 \right] dl$

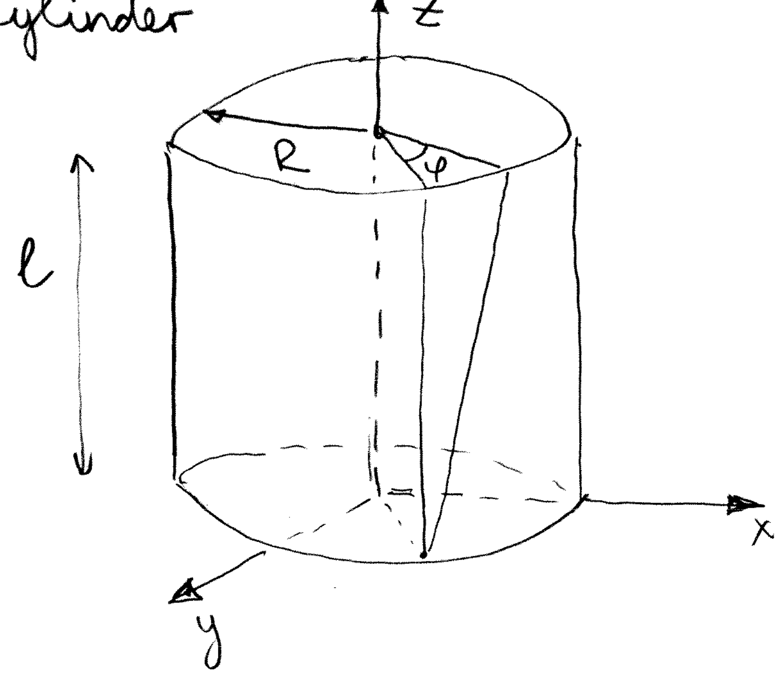
where $\vec{\tau}$ is tagential to the rod, regarded as elastic line. Those two terms correspond to the bending of the rod. For the slight bending we can approximate $\vec{\tau} = \frac{d\vec{r}}{dl} \simeq \frac{d\vec{r}}{dx}$

and $\frac{d\vec{\tau}}{dl} \simeq \left(\frac{d^2 z}{dx^2}, \frac{d^2 y}{dx^2} \right)$.

Then we arrive to the energy expression given at the p. 35.

The term $\frac{C}{2}\left(\frac{d\varphi}{dl}\xi\right)^2$ corresponds to another kind of deformation - twisting or torsion. In this case each cross-section is rotated about the axis of the rod.

Let us consider such deformation for a Cylinder



Under rotation on angle φ $\delta\vec{r} = \delta\vec{\varphi} \times \vec{r} \Rightarrow$

$$\Rightarrow \quad u_x = -y\,\varphi(z), \quad u_y = x\,\varphi(z)$$

In general u_z may not be zero, but for the cylinder one can see that $u_z = 0$

Note, that for torsion

$\operatorname{div} \vec{u} = 0 \quad \Rightarrow \quad$ pure shear deformation

Then equilibrium equation is

$$\mu \frac{\partial^2 u_i}{\partial x_k \partial x_k} = \mu \nabla^2 \vec{u} = 0$$

substituting $\quad u_x = -y\,\varphi(z), \quad u_y = x\,\varphi(z)$

we obtain

$$\frac{d^2 \varphi(z)}{dz^2} = 0 \quad \Rightarrow \quad \varphi(z) = \varphi_0 \frac{z}{\ell}$$

$$u_{xz} = -\frac{y}{2}\frac{d\varphi}{dz}, \quad u_{yz} = \frac{x}{2}\frac{d\varphi}{dz}, \quad \text{other } u_{ik} = 0$$

Then $\sigma_{xz} = 2\mu u_{xz} = -\mu y \frac{d\varphi}{dz}, \quad \sigma_{yz} = 2\mu u_{yz} = \mu x \frac{d\varphi}{dz}$

$$F = \int dz\, d^2r \left(\mu u_{ik}^2\right) = 2\mu \int \left(u_{xz}^2 + u_{yz}^2\right) d^2r\, dz =$$

$$= \mu \int dz\, d^2r \left(\frac{d\varphi}{dz}\right)^2 \frac{x^2+y^2}{2} = \frac{\pi R^4}{2}\mu \int \frac{1}{2}\left(\frac{d\varphi}{dz}\right)^2 dz$$

$= \frac{C}{2}\int \left(\frac{d\varphi}{dz}\right)^2 dz$, where the <u>torsional rigidity</u>

$$C = \frac{\pi}{2}\mu R^4$$

Assume that the bottom of the cylinder is fixed. What is the torque M that one need to apply in order to twist the top on the angle φ_0?

Energy is $\int \left[\frac{1}{2} C \left(\frac{d\varphi}{dz}\right)^2 dz + U\right.$,

where U is the energy due to external force.

varying with respect to φ we find

$$\int C \frac{d\varphi}{dz} \frac{d\delta\varphi}{dz} dz + \delta U = 0$$

integrating by parts

$$-C\int \frac{d^2\varphi}{dz^2} \delta\varphi \, dz + C \frac{d\varphi}{dz} \delta\varphi \Big|_0^l + \delta U$$

Since $\delta U = -M \delta\varphi$ we obtain

$$-C\int \frac{d^2\varphi}{dz^2} \delta\varphi dz + \varphi_0 \left[C \frac{d\varphi}{dz} - M\right] = 0$$

In the first term $\delta\varphi$ is arbitrary $\Rightarrow \frac{d^2\varphi}{dz^2} = 0$

From the last term we get

$$M = C \frac{d\varphi}{dz} = \frac{\pi}{2} \frac{\mu \varphi_0 R^4}{l}$$