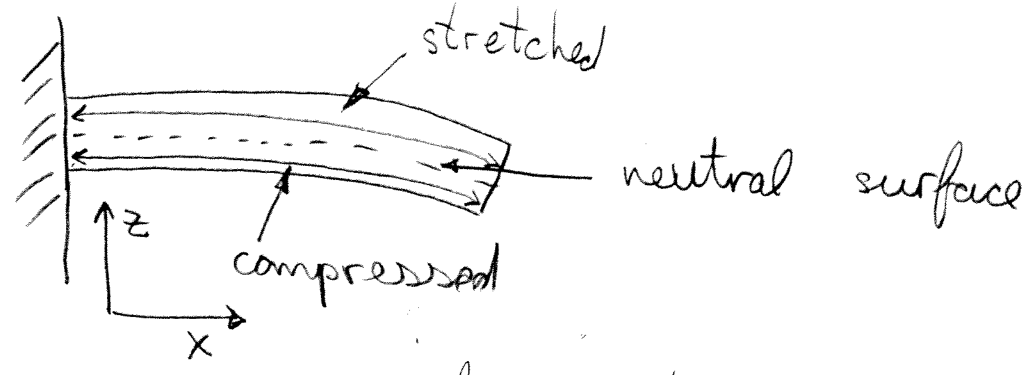
Bending of rods

A bend rod is stretched at some points and compressed at others



Since one goes from extension to compression, there is a neutral surface that separates these regions. It undergoes neither extension nor compression.

Let us consider bending deformation in a small portion of length of the rod, where bending is slight, such that the displacement of the points in the rod are small

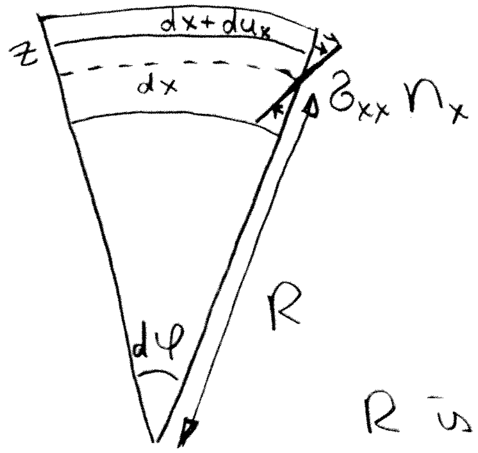
Since the rod is thin, small forces on its surface are needed to bend it. They are small compared with the internal stresses and can be taken as zero $\Rightarrow$ At the sides $\sigma_{ik} n_k = 0$

If the rod is mainly $\parallel \vec{x}$ then on its surface $n_x = 0 \Rightarrow$

$$\sigma_{zz} n_z + \sigma_{zy} n_y = 0$$

For the point of the surface where $n_y = 0$ we have $\sigma_{zz} n_z = 0 \Rightarrow \sigma_{zz} = 0$

There is another such point at the opposite side. There $\sigma_{zz} = 0$ as well. Since the rod is thin then $\sigma_{zz} = 0$ everywhere in the cross section. In the same manner one can see that all the component of the stress tensor except σ_{xx} must be zero. Thus there is a simple extension or compression in every volume element of a bend rod. The amount of this varies from point to point in every cross-section $\Rightarrow$ bending



The length of the neutral surface $dx = R\,d\varphi$ is unchanged
R is radius of curvature.

Away from the neutral surface length is changed $dx + du_x = (R + z)\,d\varphi$

where we measure z from the neutral surface $\Rightarrow$

$$du_x = z\,d\varphi$$

$$\frac{\partial u_x}{\partial x} = \frac{z\,d\varphi}{R\,d\varphi} = \frac{z}{R} \Rightarrow u_{xx} = \frac{z}{R}$$

$$\sigma_{xx} = E\,u_{xx} = \frac{E z}{R}$$

Now we should determine the position of the neutral surface. We consider pure bending, with no total extension or compression

Thus the total internal stress force on a cross-section should be zero =>

$$\int \sigma_{xx}\, dS = 0 \quad \Rightarrow \quad \int z\, dS = 0$$

From the other side center mass of the cross-section has coordinates $\frac{\int z\, dS}{\int dS}, \frac{\int y\, dS}{\int dS}$ =>

The neutral surface passes through the center mass of the cross-section of the rod.

Since for a simple extension

$$u_{zz} = u_{yy} = -\sigma u_{xx} \quad \text{then}$$

$$u_{xx} = \frac{\partial u_x}{\partial x} = \frac{z}{R}, \quad \frac{\partial u_z}{\partial z} = \frac{\partial u_y}{\partial y} = -\frac{\sigma z}{R}$$

$$u_{xz} = \frac{\partial u_z}{\partial x} + \frac{\partial u_x}{\partial z} = 0, \quad \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} = 0, \quad \frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y} = 0$$

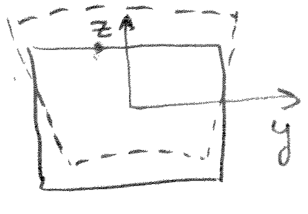
Integrating we obtain

$$u_z = -\frac{1}{2R}\left(x^2 + \sigma(z^2 - y^2)\right)$$

$$u_y = -\frac{\sigma z y}{R}$$

$$u_x = \frac{z x}{R}$$

The shape of the cross-section changes. For rectangular rod



The free energy per unit volume is

$$\frac{\sigma_{ik} u_{ik}}{2} = \frac{\sigma_{xx} u_{xx}}{2} = \frac{1}{2} E \frac{z^2}{R^2}$$

Integrating over the cross-section we have

$$\frac{1}{2} \frac{E}{R^2} \int z^2 dS = \frac{I_y E}{2R^2},$$

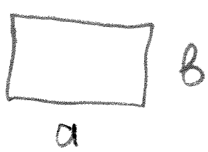
where we introduced the moment of inertia about the y-axis $I_y = \int z^2 dS$

The bending moment.

The force in the x direction is $\sigma_{xx} dS = \frac{zE}{R} dS$

It produces torque along the y axis

$$M_y = \frac{E}{R} \int z^2 dS = \frac{E I_y}{R}$$

For rectangular 

$$I_y = \frac{a b^3}{12}$$

circle 

$$I = \frac{\pi r^4}{4}$$

For the slight bending the radius of curvature $\frac{1}{R} = \pm \frac{d^2 z}{dx^2}$ and we can rewrite the energy of the rod as

$$F = \frac{1}{2} E \int I_y \left(\frac{d^2 z}{dx^2}\right)^2 dx$$

or for general bending in two possible directions

$$F = \frac{1}{2} E \int \left[I_1 \left(\frac{d^2 z}{dx^2}\right)^2 + I_2 \left(\frac{d^2 y}{dx^2}\right)^2 \right] dx$$

where I_1, I_2 are the principal moments of inertia. Note that contrary to the elastic string energy is $\propto$ to $(z'')^2$ rather than z'^2.

In the external potential $U(z,x)$

$$F = \int \left[\frac{IE}{2} (z'')^2 + U(z,x) \right] dx$$

varying with respect to z we obtain

$$\underline{I E z^{IV} = F_z}$$

One can get the same equation from the torque balance $E I_y z'' = M_y$

Examples

Rod bent by its own weight

We should solve equation

$$z^{IV} = \frac{\rho g}{IE}$$

with the appropriate boundary conditions that depend on the way of support at the ends.

There are two general cases for support:

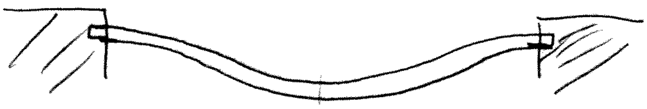
clamped  where position of the rod is fixed $\Rightarrow z = 0,\ z' = 0$ and

supported 

$\Rightarrow z = 0,\ z'' = 0$ (torque is zero)

For the clamped ends we have

$$z^{IV} = \frac{\rho g}{IE}, \quad z(0) = z'(L) = 0$$



$$Z = \frac{\rho g}{24 IE}\left(x^4 + C_1 x^3 + C_2 x^2 + C_3 x + C_4\right)$$

since $z(0) = z'(0) = 0 \Rightarrow C_3 = C_4 = 0$

$$\left.\begin{aligned} z(L) = 0 &\Rightarrow L^2 + C_1 L + C_2 = 0 \\ z'(L) = 0 &\Rightarrow 4L^2 + 3C_1 L + 2C_2 = 0 \end{aligned}\right\} \Rightarrow$$

$$C_1 = -2L, \quad C_2 = L^2 \Rightarrow$$

$$Z = \frac{\rho g}{24EI} x^2 (x-L)^2$$

In the middle

$$x\left(\frac{L}{2}\right) = \frac{\rho g L^4}{384 EI}$$

If both ends are supported =>

$z(0) = z(L) = 0$

$z''(0) = z''(L) = 0$

Then $z = \frac{\rho g}{24IE}\left(x^4 + C_1 x^3 + C_2 x\right)$

$z(L) = 0 \Rightarrow L^3 + C_1 L^2 + C_2 = 0$

$z''(L) = 0 \Rightarrow 12L + 6C_1 = 0 \Rightarrow$

$C_1 = -2L, \; C_2 = L^3 \Rightarrow$

$$z = \frac{\rho g}{24IE}\, x\left(x^3 - 2Lx^2 + L^3\right)$$

In the middle $z\left(\frac{L}{2}\right) = \frac{5\rho g L^4}{384IE}$

Clamped at one end and free at another to which a force f is applied

$EIz^{IV} = -f\,\delta(x-L) \Rightarrow$

$z''' = -\frac{f}{EI}$. At the free end torque is zero $\Rightarrow$

$\Rightarrow z''(L) = 0 \Rightarrow z = \frac{f}{6EI}x^2(3L - x)$

If the rod is under external tension one should add it to the energy =>

$$F = \int \left[\frac{EI}{2}(z'')^2 + \frac{Tz'^2}{2} + \mho\right] dx$$

In some cases tension can be result of the bending itself

In the examples discussed above bending of the rod leads to the extension of it.

The change in length of the bend rod is

$$\Delta L = \int_0^L \left(\sqrt{1+z'^2} - 1\right) dx = \frac{1}{2}\int \frac{z'^2}{2}\, dx$$

The stress force $T = E \underset{\text{cross section}}{S} \Delta L = \frac{ES}{2L}\int_0^L z'^2 dx$

If bending displacement is δ then

$T \sim \frac{ES\delta^2}{L^2}$. Compare two terms we get

$$EI(z'')^2 \sim \frac{ES^2\delta^2}{L^4}$$

$Tz'^2 \sim ES\frac{\delta^4}{L^4}$ => One should take tension into account when $\delta \sim S^{1/2} \sim r$ (thickness)