Sound

Small perturbations of density in an ideal fluid propagate as sound waves.

Assuming that the relative changes in the fluid density and pressure are small we can write

$$p = p_0 + p' \quad , \rho = \rho_0 + \rho',$$

where p_0 and ρ_0 are constant equilibrium densities and p' and ρ' are their variations in the sound wave ($p' \ll p_0$, $\rho' \ll \rho_0$).

The equation of continuity is then

$$\frac{\partial \rho'}{\partial t} + \rho_0 \operatorname{div} \vec{v} = 0$$

In the Euler's equation we can neglect nonlinear term $(v \cdot \nabla) v$ – oscillations are small $\Rightarrow$

$$\frac{\partial v}{\partial t} + \frac{1}{\rho_0} \nabla p' = 0$$

If the motion is adiabatic then we can relate p' and ρ'

$$p' = \left(\frac{\partial p}{\partial \rho}\right)_s \rho'$$

Then we have

$$\begin{cases} \dfrac{\partial \rho'}{\partial t} + \rho_0 \operatorname{div} \vec{v} = 0 \\ \rho_0 \dfrac{\partial v}{\partial t} + \left(\dfrac{\partial p}{\partial \rho}\right)_s \operatorname{grad} \rho' = 0 \end{cases}$$

$$\Downarrow$$

$$\frac{\partial^2 \rho'}{\partial t^2} - c^2 \nabla^2 \rho' = 0$$

With the sound velocity $c = \sqrt{\left(\frac{\partial p}{\partial \rho}\right)_s}$

Analogously introducing potential

$v = \operatorname{grad} \varphi \implies$

$$p' = -\rho_0 \frac{\partial \varphi}{\partial t} = \left(\frac{\partial p}{\partial \rho}\right)_s \rho'$$

Substituting it back to the continuity equation we obtain the same wave equation for the potential

$$\frac{\partial^2 \varphi}{\partial t^2} - c^2 \nabla^2 \varphi = 0$$

as well as for velocity

$$\frac{\partial^2 v}{\partial t^2} - c^2 \nabla^2 \vec{v} = 0$$

and pressure.

This wave equation leads to linear dispersion relation $\omega = c k$

If all quantities depend on only one coordinate then solution of the wave equation is

$$\varphi = f_1(x - ct) + f_2(x + ct)$$

wave propagates along x. Since $v = \operatorname{grad} \varphi$ then only $v_x = \frac{\partial \varphi}{\partial x}$ is nonzero $\Rightarrow \vec{v} \parallel \vec{k} \Rightarrow$ Sound waves in a fluid are longitudinal

Putting $\varphi = f(x-ct)$ we find

$$v = \frac{\partial \varphi}{\partial x} = f'(x-ct)$$

$$p' = -\rho_0 \frac{\partial \varphi}{\partial t} = \rho_0 c f'(x-ct)$$

Thus $v = p'/\rho_0 c$

But since $p' = c^2 \rho'$ we find the relation between the velocity and density variation

$$v = c \frac{\rho'}{\rho_0}$$

Thus small amplitude $\rho' \ll \rho_0$ means $v \ll c$. The pressure variation in the sound wave $p' = \rho v c$ is much larger than ρv^2 in an incompressible flow.

Note, that the sound velocity is given by the adiabatic compressibility $c^2 = \left(\frac{\partial p}{\partial \rho}\right)_S$ which is different from the isothermal one

$$\left(\frac{\partial p}{\partial \rho}\right)_S = \gamma \left(\frac{\partial p}{\partial \rho}\right)_T , \quad \gamma = \frac{c_p}{c_v}$$

For an ideal diatomic gas (air) $c_v = \frac{5}{2}$

$$c_p - c_v = 1 \Rightarrow \gamma = \frac{7}{5}$$

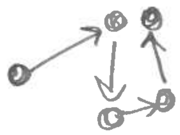
$$pV = \frac{RT}{M} \Rightarrow c = \left(\gamma \frac{RT}{M}\right)^{1/2} =$$

Sound velocity $\propto \sqrt{T}$ and does not depend on pressure at given temperature.

Newton knew that $c^2 = \frac{\partial p}{\partial \rho}$. Experimental data from Boyle showed $p \propto \rho$ (i.e. they were isothermic) so one gets $c = \sqrt{\frac{p}{\rho}} \simeq 290 \frac{m}{s}$ well of observed value $c \simeq 340$ m/s at 20°C. Only hundred years later Laplace got the true (adiabatic) value with $\gamma = \frac{7}{5}$.

Adiabatic motion in the sound wave means, that the heat of the compression does not have time to escape the pressure pulse.

Molecules moves ballistically with thermal velocity v_{th} over the mean free path ℓ



Then their motion becomes diffusive with $\langle R^2 \rangle \simeq v_{th} \ell t$

for $t > \tau \simeq \dfrac{\ell}{v_{th}}$

Thermal equilibration is due to this diffusive motion of molecules. It is slow if during the period of oscillation in the wave typical displacement of molecules is much less than the wavelength $\langle R^2(T) \rangle \ll \lambda^2 \Rightarrow$

$v_{th} \ell T \ll \lambda^2 \Rightarrow v_{th} \ell \ll c \lambda$

Sound velocity $c \simeq v_{th} \Rightarrow$ motion is adiabatic if $\underline{\ell \ll \lambda}$

Spherical wave

For spherical symmetry we have

$$\frac{\partial^2 \varphi}{\partial t^2} = c^2 \frac{1}{r^2} \frac{\partial}{\partial r}\left(r^2 \frac{\partial \varphi}{\partial r} \right)$$

Substituting $\varphi = \frac{f(r,t)}{r}$ we obtain

$$\frac{\partial^2 f}{\partial t^2} = c^2 \frac{\partial^2 f}{\partial r^2}$$

Thus the general solution is

$$\varphi = \frac{f_1(r-ct)}{r} + \frac{f_2(r+ct)}{c}$$

Unlike a plane wave here amplitude decreases as $\frac{1}{r}$, the intensity $\sim \frac{1}{r^2}$

Consider outgoing spherical wave. Before the wave arrives $\varphi = 0$. After the wave has passed $\varphi \to$ const. But $\varphi = \frac{f(ct-r)}{r} \Rightarrow f = 0 \Rightarrow \varphi = 0$

$p' = -\rho_0 \frac{\partial \varphi}{\partial t} \Rightarrow \int_{-\infty}^{\infty} p' \, dt = 0 \Rightarrow$ as the spherical wave passes both condensation ($p' > 0$) and rarefactions ($p' < 0$) will be observed. This is not the case for a plane wave.

Propagation of sound in a moving medium

The relation $\omega = ck$ is valid only for a monochromatic sound wave propagated in a medium at rest. Consider homogeneous flow with velocity $\vec{u}$. We take a fixed coordinate (x, y, z) system is K and a system K' moving with the fluid with velocity $\vec{u}$. In the system K' fluid is at rest and a monochromatic wave has form $\varphi \propto \exp(i\vec{k}\cdot\vec{r}' - kct)$.

Since $r' = r - ut$, then in the fixed system the wave has the form $\varphi \propto \exp(i\vec{k}\cdot\vec{r} - kct - \vec{k}\cdot\vec{u}t)$

Then the frequency in a moving media is

$$\omega = c|k| + \vec{u}\cdot\vec{k}$$

The velocity of propagation is

$$\frac{\partial \omega}{\partial \vec{k}} = \frac{c\vec{k}}{k} + \vec{u}$$

Doppler effect

The frequency of sound as received by an observer moving relative to the source is not the same as frequency of oscillation of the source.

Let sound emitted by a source at rest be received by an observer moving with velocity $\vec{u}$ (relative to the medium). In a system K' of the ~~source~~ medium is at rest and $k = \frac{\omega_0}{c}$. In a system moving with observer fluid has velocity $-\vec{u}$ and the sound frequency is

$$\omega = ck - \vec{u} \cdot \vec{k}$$

Introducing the angle θ between $\vec{u}$ and $\vec{k}$ and using $k = \omega_0 / c$ we obtain

$$\omega = \omega_0 \left[1 - \frac{u}{c} \cos\theta \right]$$

The opposite case is when the source of sound is moving with respect to a medium

Let $\vec{u}$ be the velocity of the source.
K' is the system moving with that source
In that system fluid moves with velocity $-\vec{u}$.
In K' the source is at rest and frequency is ω_0.

Then $\omega_0 = ck\left[1-\frac{u}{c}\cos\theta\right]$. In the original laboratory frame K $\omega = ck \Rightarrow$

$$\omega = \frac{\omega_0}{1-\frac{u}{c}\cos\theta}$$

If the source moves away from the observer

$\frac{\pi}{2} < \theta \leq \pi \Rightarrow \cos\theta < 0 \Rightarrow$ frequency heard (ω) is less than emitted $\omega < \omega_0$.

If the source is approaching observer

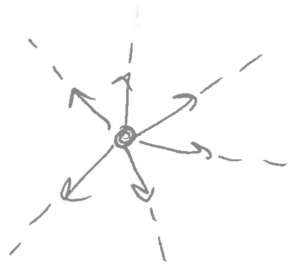
$0 \leq \theta < \frac{\pi}{2} \Rightarrow \cos\theta > 0$ and $\omega > \omega_0$

For $u\cos\theta > c$ ω becomes negative, which means that the sound heard by observer reachs him in reversed order. The emitted later sound arrives earlier.

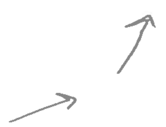
Why we don't hear if some one is shouting against the wind?

Wind velocity $u \ll c \Rightarrow$ sound is not being carried by the wind. It is actually refracted because of wind gradient.

<u>Geometric acoustics</u>. For $r \gg \frac{1}{k}$ we can introduce sound rays. In homogeneous medium rays are propagated in straight lines.



If the medium moves $u_x = u(z)$, $u_y, u_z = 0$, then the rays direction is changed. If $\vec{n}$ is the unit vector along the propagation, then during time dt $dn_x = \frac{du_x}{c}$, $\delta n_y = 0$ $\Rightarrow$



$$\frac{dn_x}{dt} = \frac{du_x}{c\,dt} \Rightarrow n_x = n_{x0} + \frac{u(z)}{c}$$

$n_x^2 + n_z^2 = 1$