# Group velocity

Consider wave packet which occupies some finite region of space. Assume it is almost monochromatic

$$\varphi = e^{ikr} f(r)$$

where $f(r) = \sum_{q \ll K} f_q e^{iqr}$ — slowly (compare to $1/K$) varying function

Then after time $t$

$$\varphi(r,t) = \sum_q e^{ikr + iqr - i\omega(k+q)t} f_q =$$

$$= e^{ikr - i\omega(k)t} \sum f_q e^{iqr - i\frac{\partial\omega}{\partial\vec{K}}\cdot q t} =$$

$$= e^{ikr - i\omega(k)t} \sum f_q e^{i\vec{q}(\vec{r} - i\frac{\partial\omega}{\partial\vec{K}}t)} =$$

$$= e^{ikr - i\omega(k)t} f\left(r - \frac{\partial\omega}{\partial\vec{k}}t\right)$$

Thus the amplitude distribution moves by $\frac{\partial\omega}{\partial\vec{K}}t$

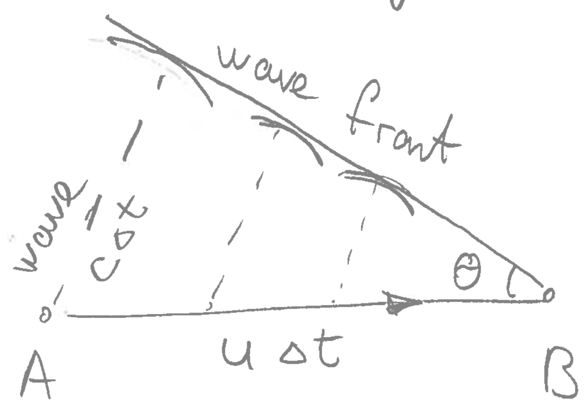
$\frac{\partial\omega}{\partial\vec{K}}$ is called the group velocity

$\frac{\omega}{K}$ — phase velocity

# Ship wave

Nonlinear dispersion leads to the change of the wave shape.

If the body moves in the media with the supersonic velocity $u > c$ then the Mach cone is formed




$$\sin \Theta = \frac{c}{u}$$

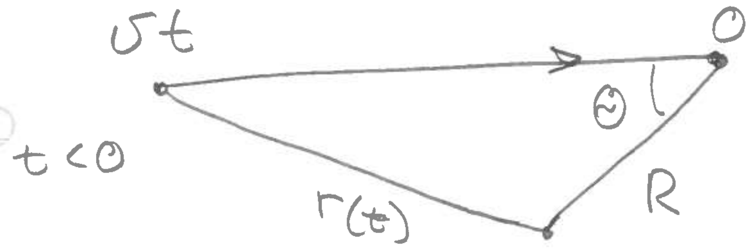
For $u \gg c$ $\Theta \to 0$

Waves behind the ship are moving with different velocities that depend on the wave vector $v_{gr} = \frac{d\omega}{dk} = \frac{1}{2}\sqrt{\frac{g}{k}}$

We should sum over all possible "rings" produced by the ship during its motion

$$h(t) \propto \int_{-\infty}^{0} h_t \, dt$$

with $h_t \propto \exp(i\, u(t))$ , $u = \dfrac{g t^2}{4 r(t)}$




$$r(t) = \sqrt{R^2 + v^2 t^2 + 2 R v t \cos\theta} \qquad t < 0$$

Using method of stationary phase

$$\dot{u}(t) = \frac{g}{4}\left\{ \frac{2t}{r(t)} - \frac{t^2 (v^2 t + R v \cos\theta)}{r^3(t)} \right\} =$$

$$= \frac{g t}{4 r^3}\left[ 2r^2 - v^2 t^2 - R v t \cos\theta \right] =$$

$$= \frac{g t}{4 r^3}\left( v^2 t^2 + 3 R v t \cos\theta + 2R^2 \right)$$

Equating $\dot{u}(t) = 0$ we obtain

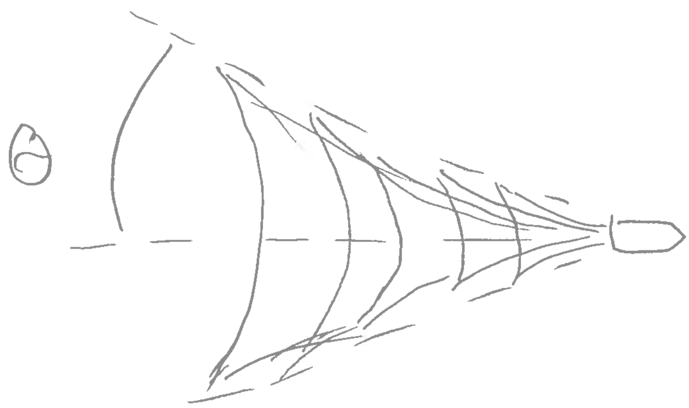
$$t = -\frac{3R}{v}\left\{\cos\theta \pm \sqrt{\cos^2\theta - \frac{8}{9}}\right\} =$$

$$= -\frac{3R}{v}\left\{\cos\theta \pm \sqrt{\frac{1}{9} - \sin^2\theta}\right\}$$

These roots are real (and negative)

for $\sin\theta < \sin\theta_0 = \frac{1}{3} \Rightarrow$

$\theta < \theta_0 = 19.28^\circ$




Note that this Kelvin angle does not depend on the ship velocity!

# Capillary waves

Molecules of liquid close to the surface feel different surrounding than those deeply inside. In general their energy is higher. Thus one has to pay for existance of the surface – surface energy. When one deforms the surface its area is changed and the change of energy $\delta R = \alpha\, \delta S$ where $\delta S$ is change of area and $\alpha$ is surface tension coefficient.

Deformation of the surface leads to additional pressure. Indeed $\nabla \xi = \left(\frac{\partial \xi}{\partial x}, \frac{\partial \xi}{\partial y}\right)$

$$S = \int dx\, dy \sqrt{1+(\nabla\xi)^2} = \int d^2 r \left(1 + \frac{(\nabla \xi)^2}{2}\right)$$

$$\delta S = \int d^2 r\ \nabla \xi \nabla \delta \xi = -\int d^2 r\ \nabla^2 \xi \cdot \delta \xi$$

Change of energy $\alpha \delta S$ should be balanced by pressure

$$\alpha \delta S - \int p \, \delta \xi \, dS = 0 \quad \Rightarrow$$

$$p = -\alpha \left( \frac{\partial^2 \xi}{\partial x^2} + \frac{\partial^2 \xi}{\partial y^2} \right)$$

This pressure should be included to the equation we used in deriving gravity waves

$$P = -\rho \left( g \xi + \rho \frac{\partial \varphi}{\partial t} \right) \quad \Rightarrow$$

$$\rho g \xi + \rho \frac{\partial \varphi}{\partial t} - \alpha \left( \frac{\partial^2 \xi}{\partial x^2} + \frac{\partial^2 \xi}{\partial y^2} \right) = 0$$

Taking time derivative and replacing

$$\frac{\partial \xi}{\partial t} = v_z = \frac{\partial \varphi}{\partial z}$$ we arrive

$$\left[ \rho g \frac{\partial \varphi}{\partial z} + \rho \frac{\partial^2 \varphi}{\partial t^2} - \alpha \frac{\partial}{\partial z} \left( \frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} \right) \right]_{z=0} = 0$$

Looking for solution

$$\varphi = A e^{kz} \cos(kx - \omega t)$$

we obtain dispersion relation for gravity-capillary waves: $\omega^2 = gk + \frac{\alpha}{\rho} k^3$

For $k \gg k_* = \sqrt{\frac{\rho g}{\alpha}}$ we have capillary waves with $\omega^2 = \frac{\alpha}{\rho} k^3$

For water this corresponds to $\lambda \ll \lambda_* \simeq 1.6\,cm$.

Why does water pour out of overturned glass?

The atmospheric pressure can hold a water column as high as 10 meters.

The reason is that with negative gravity we should change sign of $g$ in dispersion relation $\Rightarrow$

$$\omega(k) = \sqrt{-gk + \frac{\alpha}{\rho} k^3}$$

For small $k$ frequency is imaginary $\Rightarrow$ surface is unstable with respect to the ripple formation (Rayleigh-Taylor instability). If diameter is smaller than $\lambda^*$ the surface is stable - capillary.