Energy dissipation in an incompressible fluid

Kinetic energy is

$$E_{kin} = \frac{1}{2}\rho\int v^2 dV$$

$$\frac{\partial}{\partial t}\rho\frac{v^2}{2} = \rho\,\vec{v}\cdot\frac{\partial\vec{v}}{\partial t}$$

Substituting $\frac{\partial\vec{v}}{\partial t}$ from the Navier - Stokes Eq.

$$\frac{\partial}{\partial t}\left(\rho\frac{v^2}{2}\right) = -\rho\vec{v}\cdot(\vec{v}\cdot\vec{\nabla})\vec{v} - \vec{v}\cdot\vec{\nabla}p + v_i\frac{\partial\sigma'_{ik}}{\partial x_k} =$$

$$= -\mathrm{div}\left[\rho\vec{v}\left(\frac{v^2}{2}+\frac{p}{\rho}\right) - \vec{v}\cdot\sigma'\right] - \sigma'_{ik}\frac{\partial v_i}{\partial x_k}$$

We used the fact that $\mathrm{div}\,\vec{v} = 0$.

Here $\vec{v}\cdot\sigma' = v_i\sigma'_{ik}$

The first term in the brackets [ ] is just the energy flux in the ideal liquid due to the actual transfer of fluid mass.

The second term, $\vec{v}\cdot\sigma'$ is the energy flux due to processes of internal friction.

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The energy flux is equal to the scalar... product of the momentum flux σ'_{ik} and the velocity.

Integrating over volume we obtain

$$\frac{\partial}{\partial t}\int \rho\frac{v^2}{2}\,dV = -\oint\left[\rho\vec{v}\left(\frac{v^2}{2}+\frac{P}{\rho}\right)-\vec{v}\cdot\sigma'\right]dS - \int \sigma'_{ik}\frac{\partial v_i}{\partial x_k}d$$

Extending integration to the whole volume the surface term vanishes and

$$\frac{dE_{kin}}{dt} = -\int \sigma'_{ik}\frac{\partial v_i}{\partial x_k} = -\frac{1}{2}\int \sigma'_{ik}\left(\frac{\partial v_i}{\partial x_k}+\frac{\partial v_k}{\partial x_i}\right)dV =$$

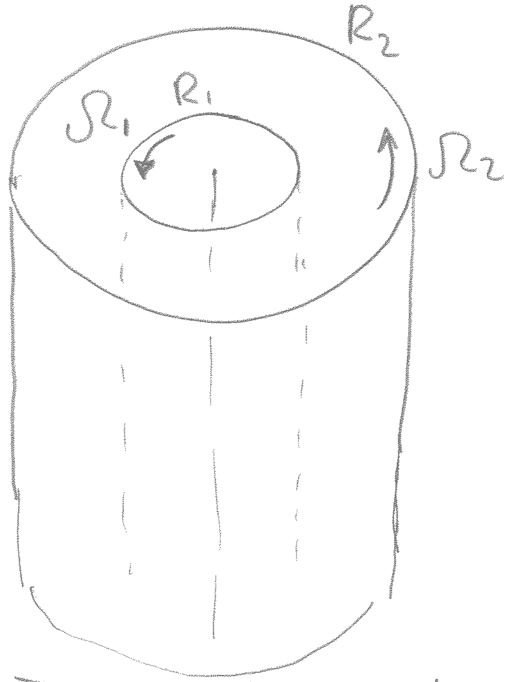
$$= -\frac{1}{2}\eta\int\left(\frac{\partial v_i}{\partial x_k}+\frac{\partial v_k}{\partial x_i}\right)^2 dV$$

Dissipation leads to a decrease of energy =>

$$\frac{dE_{kin}}{dt} < 0 \quad \Rightarrow \quad \eta > 0$$

Flow between rotating cylinders

Couette flow (M. Couette 1890)



In cylindrical coordinates

$$v_z = v_r = 0\,, \quad v_\varphi = v(r)\,, \quad p = p(r)$$

$$\frac{\partial \vec{e}_\varphi}{\partial \varphi} = -\vec{e}_r$$

In this situation $(\vec{v}\cdot\vec{\nabla})\vec{v} = \frac{v(r)}{r}\frac{\partial}{\partial\varphi} v(r)\vec{e}_\varphi = -\frac{v^2(r)}{r}\vec{e}_r$

in the Navier-Stokes equation we have

$$\frac{dp}{dr} = \frac{\rho v^2}{r}\,, \qquad \eta \nabla^2 v \Big|_\varphi = 0$$

$$\frac{d^2 v}{dr^2} + \frac{1}{r}\frac{dv}{dr} - \frac{v}{r^2} = 0 \qquad \frac{\partial^2 \vec{e}_\varphi}{\partial \varphi^2} = -\vec{e}_\varphi$$

Looking for solution of the form r^n

we get $n(n-1) + n - 1 = 0 \Rightarrow$

$n^2 = 1 \Rightarrow \quad n = \pm 1 \Rightarrow$

$$v = ar + \frac{b}{r}$$

The constants a and b are found from the boundary conditions $v(R_1) = \Omega_1 R_1$, $v(R_2) = \Omega_2 R_2$. As a result

$$v = \frac{\Omega_2 R_2^2 - \Omega_1 R_1^2}{R_2^2 - R_1^2}\, r + \frac{(\Omega_1 - \Omega_2) R_1^2 R_2^2}{R_2^2 - R_1^2}\, \frac{1}{r}$$

- For $\Omega_1 = \Omega_2 = \Omega$ we have

$$v = \Omega r \implies$$ fluid rotates rigidly with the cylinders

- If outer cylinder is absent $\Omega_2 = 0$, $R_2 = \infty$ then

$$v = \frac{\Omega_1 R_1^2}{r}$$

Note that equation for velocity does not depend on viscosity

Let us determine the moment of the frictional forces acting on the cylinders. The frictional force acting on unit area of the inner cylinder is equal to the component $\sigma'_{r\varphi}$ of the stress tensor, $f_i = -\sigma'_{ik} n_k$

outward normal to the fluid

$$[\sigma'_{r\varphi}]_{r=R_1} = \eta \left[\frac{\partial v}{\partial r} - \frac{v}{r}\right]_{r=R_1} =$$

$$= -2\eta \frac{(\Omega_1 - \Omega_2) R_2^2}{R_2^2 - R_1^2}$$

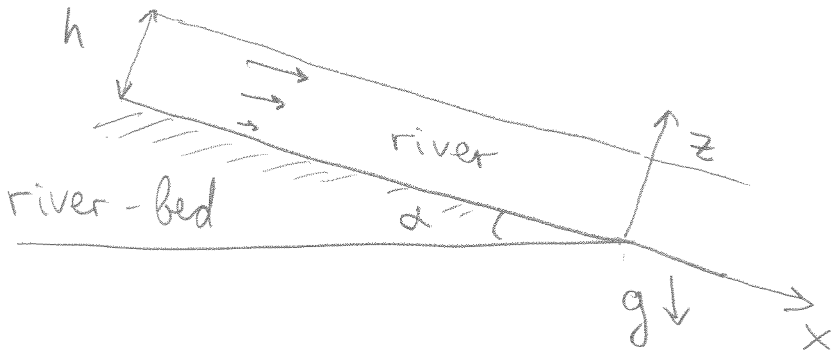
The moment of this force is found by multiplying by R_1 and the total moment M_1 acting on unit length of the cylinder by multiplying by $2\pi R_1$

Thus $$M_1 = -\frac{4\pi\eta (\Omega_1 - \Omega_2) R_1^2 R_2^2}{R_2^2 - R_1^2}$$

The torque acting on the outer cylinder

$$M_2 = -M_1$$

River flow



In this geometry $(\vec{v}\cdot\vec{\nabla})\,\vec{v} = 0 \Rightarrow$

- Navier - Stokes equation is

$$-\vec{\nabla} p + \eta \nabla^2 \vec{v} + \rho \vec{g} = 0$$

$$v_x = v(z)\,,\ v_y = v_z = 0\,,\ p = p(z) \Rightarrow$$

$$\frac{dp}{dz} + \rho g \cos\alpha = 0$$

- $$\eta \frac{d^2 v}{dz^2} + \rho g \sin\alpha = 0$$

Boundary condition at the bottom is $v(0) = 0$.

On the surface the boundary condition is that the stress should be normal and balance the atmospheric pressure p_0

$$\sigma_{xz}(h) = \eta \frac{dv}{dz} = 0 \quad , \sigma_{zz} = -p(h) = -p_0$$

The solution is

$$p(z) = p_0 + \rho g (h-z) \cos\alpha$$

$$v(z) = \frac{\rho g \sin\alpha}{2\eta} z (2h-z)$$

Let's see how it corresponds to reality

For water $\nu = \frac{\eta}{\rho} \simeq 10^{-2} \frac{cm^2}{sec}$

For a rain paddle with the thickness $h = 1mm$ on a slope $\alpha \sim 10^{-2}$ we get a reasonable estimate $v \sim 5\ cm/sec$

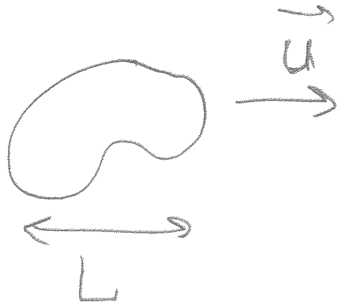
For slow plain river with $h \simeq 10m$ and $\alpha \simeq 0.1km/1000km \simeq 10^{-4}$ one gets

$v(h) \simeq 100 \frac{km}{sec}$ which is evidently impossible (real rivers are turbulent)

What is the difference between paddle and river?

The law of similarity

Consider some particular kind of stationary motion of the body of definite shape in a fluid



There are the following parameters that velocity $v(r)$ depends on: body velocity u, size of the body L (for given shape) and viscosity ν. Their dimensions are

$$[u] = \frac{cm}{sec}, \quad [L] = cm, \quad \nu = \frac{cm^2}{sec}$$

One can make only one dimensionless combination out of them

$$Re = \frac{uL}{\nu}$$ <u>Reynolds number</u>

Dimensionless velocity must be a function of dimensionless variables:

$$v(r) = u\, f\left(\frac{r}{L}, Re\right)$$

Flows that corresponds to the same Re can be obtained from one another by changing the units of v and r (rescaling), such flow are called <u>similar</u> (Reynolds 1883). This law of similarity is exploited in modeling: to measure, say, a drag on the ship one designs, one can build a smaller model yet pull it faster through the fluid or use less viscous fluid. The physical meaning of the Reynolds number is that it determines the ratio of the nonlinear (inertial) term $\rho(\vec{v}\cdot\vec{\nabla})\vec{v}$ to the viscous term $\eta\nabla^2\vec{v}$.

Note that for the inclined plane flow the nonlinear term (and the Reynolds number) is identically zero since $\vec{v} \perp \vec{\nabla} v$. How much one needs to perturb this alignment to make $Re \simeq 1$? Denoting $90^\circ - \beta$ the angle between v and ∇v we get $Re(\beta) = \frac{v(h) h \beta}{\nu} \simeq \frac{g \alpha h^3 \beta}{\nu^2}$

- For a rain paddle $Re(\beta) \simeq 100\beta$ while for a river $Re(\beta) \simeq 10^{12}\beta$

It is clear that the laminar solution found above may make sense for a paddle

- But for a river it must be unstable to even tiny vialation of this symmetry

Dimensional estimates

Let us estimates the drag force for small velocity. If $Re \ll 1$ then we can drop inertial term $\rho(v\cdot\nabla)v$ in the Navier-Stokes equation. Then the viscous force can depend only on η, L, v.

- Since $[\eta] = \frac{g}{cm \cdot sec}$ and dimension of force is $[F] = \frac{g \cdot cm}{sec^2} \Rightarrow F \sim \eta v \cdot L$

  As we will see for sphere of radius R
  $F = 6\pi\eta v R$ (Stokes law)

- For large velocity in the turbulent regime viscosity drops out, then the force depends only on ρ, L, v and $F \sim \rho v^2 L^2$

In general $F = \eta v L f\Big(\underbrace{\frac{vL\rho}{\eta}}_{Re}\Big)$

Transition to turbulent flow happens at $Re \sim 100$