Conservation laws and simple flows

The conservation of circulation

The velocity circulation around the contour C is defined as

$$\Gamma = \oint_C \vec{v} \cdot d\vec{l}$$

Let us calculate its time derivative

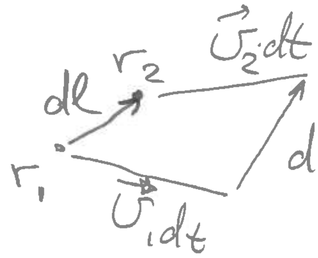
$$\frac{d}{dt} \oint \vec{v} \cdot d\vec{l}$$

Here we consider the total time derivative $\Rightarrow$ the change in the circulation around a "fluid contour" that moves with the fluid

In taking time derivative we should take into account not only change of velocity with time but also change of the contour C.

$$\frac{d}{dt}\oint \vec{v}\cdot d\vec{l} = \oint \frac{d\vec{v}}{dt}\cdot d\vec{l} + \oint \vec{v}\cdot \frac{d(dl)}{dt}$$

In the last integral



$$\vec{dl}' = \vec{r}_2 + \vec{v}(r_2)\,dt - \vec{r}_1 - \vec{v}(r_1)\,dt =$$

$$= dl + dt\cdot(\vec{dl}\cdot\vec{\nabla})\vec{v}$$

Thus $\frac{d}{dt}dl = (\vec{dl}\cdot\vec{\nabla})\vec{v}$

As a result

$$\oint \vec{v}\cdot\frac{d}{dt}(dl) = \oint \vec{v}\cdot(\vec{dl}\cdot\vec{\nabla})\vec{v} = \underline{\oint dl\,\nabla\frac{v^2}{2} = 0}$$

Then $\frac{d\Gamma}{dt} = \oint \frac{d\vec{v}}{dt}\,dl$

From the Euler's equation

$$\frac{d\vec{v}}{dt} = -\operatorname{grad} W \Rightarrow \frac{d\Gamma}{dt} = -\oint \nabla W\,dl = 0 \Rightarrow$$

$$\boxed{\oint \vec{v}\,dt = \text{const}}$$

Conservation of circulation (Kelvin's theorem 1869)

Using Stokes theorem for an infinitesimal closed contour we can rewrite

$$\oint \vec{v} \cdot dl = \int \mathrm{rot}\, \vec{v} \cdot d\vec{s} \simeq \delta \vec{S} \cdot \mathrm{rot}\, \vec{v} = \mathrm{const}$$

Here $\delta \vec{S}$ is a fluid surface element spanning the contour and $\mathrm{rot}\, \vec{v} = \vec{\Omega}$ is vorticity. For incompressible fluid area is unchanged. Than one can interprete $\delta \vec{S} \cdot \mathrm{rot}\, \vec{v} = \mathrm{const}$ as the statement that vorticity moves with the fluid (Helmholtz theorem)

More rigorously, the distance between two fluid particles satisfies equation

$$\frac{d\vec{r}}{dt} = (\vec{r} \cdot \vec{\nabla})\, \vec{v}$$

We will see that the vorticity $\vec{\Omega}$ satisfies the same equation

Indeed, taking rot of the Euler's equation we obtain

$$\frac{\partial \vec{\Omega}}{\partial t} = \mathrm{rot}[\vec{v} \times \vec{\Omega}]$$

Using $\vec{\nabla} \times (\vec{A} \times \vec{B}) = \vec{A} \cdot (\vec{\nabla} \cdot \vec{B}) - \vec{B} \cdot (\vec{\nabla} \cdot \vec{A}) + \vec{B} (\vec{\nabla} \vec{A}) - A (\vec{\nabla} \cdot \vec{B})$

we obtain

$$\mathrm{rot}\,(\vec{v} \times \vec{\Omega}) = (\vec{\Omega} \cdot \vec{\nabla}) \vec{v} - (\vec{v} \cdot \vec{\nabla}) \vec{\Omega} - \vec{\omega} (\vec{\nabla} \cdot \vec{v}) + \vec{v} \cdot (\nabla \cdot \vec{\Omega})$$

Since for incompressible fluid $\vec{\nabla} \cdot \vec{v} = 0$ and $\vec{\nabla} \cdot \vec{\Omega} = \mathrm{div}\,\mathrm{rot}\, v = 0$ as well, we arrive at

$$\frac{\partial \Omega}{\partial t} + (v \cdot \nabla) \Omega = (\Omega \cdot \nabla) \vec{v} \quad \text{or}$$

$$\frac{d \vec{\Omega}}{d t} = (\vec{\Omega} \cdot \vec{\nabla}) \vec{v}$$

Which is the same equation as one for the distance between two fluid particles.

Vortex lines

For homogeneously rotated fluid

$\Omega = \mathrm{rot}\, v = \mathrm{const} \Rightarrow v = \frac{[\vec{\Omega} \times \vec{r}]}{2}$

For inhomogeneous situation one can use magnetic analogy. $\vec{B} = \mathrm{rot}\, \vec{A}$, $\mathrm{div}\, \vec{B} = 0$

Equation for vorticity is the same $\Rightarrow$ lines of field $\vec{\Omega}$ are called vortex line

One can consider individual vortex line as velocity field that satisfies

$\mathrm{rot}\, v = \Omega_0\, \delta^2(r)$

For incompressible fluid $\mathrm{div}\, \vec{v} = 0 \Rightarrow$

$$\vec{v} = \frac{[\vec{\Omega} \times \vec{r}]}{2\pi r^2}$$

This divergent velocity field should be cut off at the distance $\underline{a}$ which is called <u>vortex core</u> radius.

Incompressible and irrotational flows

What are conditions for incompressible flow?

When the pressure changes adiabatically

$$\Delta \rho = \left(\frac{\partial \rho}{\partial p}\right)_s \Delta p = \frac{\Delta p}{c^2}$$

Where the sound velocity $c = \sqrt{\left(\frac{\partial p}{\partial \rho}\right)_s}$

Indeed from the longitudinal sound in solids we have $c_e = \sqrt{\frac{K}{\rho}}$

Energy is $\frac{K(\operatorname{div} u)^2}{2} = \frac{K}{2}\left(\frac{\delta V}{V}\right)^2 \Rightarrow$

$$K = V^2\left(\frac{\partial^2 E}{\partial V^2}\right)_s = -V^2\left(\frac{\partial p}{\partial V}\right)_s = \rho\left(\frac{\partial p}{\partial \rho}\right)_s \Rightarrow$$

$$c^2 = \frac{K}{\rho} = \left(\frac{\partial p}{\partial \rho}\right)_s$$

According to Bernoulli's equation

$$\Delta p \sim \rho v^2 \Rightarrow \Delta \rho = \frac{\Delta p}{c^2} \sim \rho \frac{v^2}{c^2}$$

Thus $\frac{\delta\rho}{\rho} \ll 1$ is equivalent to $\underline{v \ll c} \Rightarrow$ fluid velocity should be much less than the sound velocity.

In nonsteady flow we should estimate

$\frac{\partial\rho}{\partial t} \simeq \frac{\delta\rho}{\tau}$ where τ is the typical time scale of velocity change (ℓ - length scale)

Then $\frac{\partial\rho}{\partial t} \simeq \frac{\delta\rho}{\tau} \sim \frac{\delta P}{\tau c^2} \sim \frac{\rho v \ell}{\tau^2 c^2}$

In the last estimate we used $\frac{\partial v}{\partial t} \sim \frac{\nabla P}{\rho}$.

$\frac{\partial\rho}{\partial t}$ can be neglected compared with $\rho \operatorname{div} v \sim \rho \frac{v}{\ell}$ in the continuity equation if

$$\underline{\tau \gg \ell/c}$$

Since sound equilibrates densities in different points all processes must be slow to let sound pass.

For irrotational flow ($\Leftrightarrow$ potential flow)

$\text{rot}\, \vec{v} = 0$

everywhere in the space

Then we can define velocity potential

$$\vec{v} = \text{grad}\, \varphi$$

The Euler's equation can be rewritten as

$$\text{grad} \left(\frac{\partial \varphi}{\partial t} + \frac{v^2}{2} + W \right) = 0 \Rightarrow$$

$$\frac{\partial \varphi}{\partial t} + \frac{v^2}{2} + W = f(t)$$

$f(t)$ is an arbitrary function of time that can be absorbed in the potential $\varphi \Rightarrow$

$$\frac{\partial \varphi}{\partial t} + \frac{v^2}{2} + W = 0$$

Bernoulli's equation is then for steady flow

$$\frac{v^2}{2} + W = \text{const}$$

where const is the same not only along the streamline but everywhere throughout the fluid

Absence of vorticity provides a huge simplification. Unfortunately irrotational flows are less frequent as one would guess.

The main reason is that the viscous boundary layers near solid boundaries generate vorticity as we will see later. Still large portion of fluid often can be described as irrotational.

Another example – small oscillations (waves or small oscillation of an immersed body).

Then we can neglect the nonlinear term $(v\cdot\nabla)v$ and the Euler's equation is

$$\frac{\partial v}{\partial t} = -\nabla W$$

Taking curl we see that Ω is conserved ~~but its average is zero~~ in oscillating motion $\Rightarrow$ $\mathrm{rot}\, v = 0$

For potential flow of incompressible fluid

div $v = 0$, rot $v = 0$ $\Rightarrow$ $v = \text{grad}\,\varphi$

and

$$\nabla^2 \varphi = 0$$

Laplace equation (derived by Euler before Laplace)

Boundary condition at fixed solid surfaces

is $v_n = 0 \Rightarrow \frac{\partial \varphi}{\partial n} = 0$

Solving Laplace's equation for φ we can use electrostatic analogy.

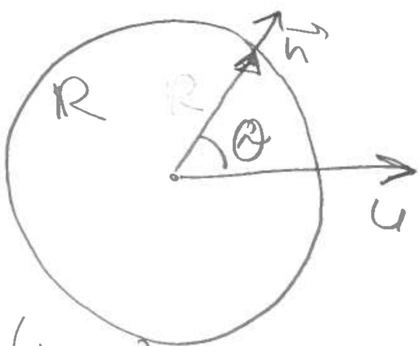
Consider for example sphere moving in a fluid with velocity $\vec{u}$

Solutions of Laplace's equation that vanishes at infinity are $\frac{1}{r}$ and $\frac{\partial^n}{\partial x^n}\left(\frac{1}{r}\right)$

Because of the complete symmetry of the sphere it is characterized by a single vetor $\vec{u}$ of its velocity. Linearity requires that $\varphi \propto \vec{u}$ and the only scalar one can make out of $\vec{u}$ and derivatives of $\frac{1}{r}$ is

$$\varphi = A\left(u \cdot \nabla \frac{1}{r}\right) = -\frac{A(\vec{u} \cdot \vec{n})}{r^2} \quad \text{(dipolar)}$$

where $\vec{n} = \frac{\vec{r}}{r}$



$$\vec{v} = \nabla \varphi = A\frac{\left(3\vec{n}(\vec{u} \cdot n) - \vec{u}\right)}{r^3}$$

At the surface of the sphere $(r = R)$

$\vec{v} \cdot \vec{n} = \vec{u} \cdot \vec{n} = u \cos\theta$ then multiplying by $\vec{n}$

we get $\quad A = \frac{R^3}{2} \Rightarrow$

$$\varphi = -\frac{R^3}{2r^2}(\vec{u} \cdot \vec{n}) \quad \text{(Stokes 1843)}$$

$$\vec{v} = \frac{R^3 \left(3\vec{n}(\vec{u} \cdot \vec{n}) - \vec{u}\right)}{2r^3}$$

Since for uncompressible fluid we can replace w by $\frac{p}{\rho}$ then pressure can be calculated from

$$p = p_0 - \frac{\rho v^2}{2} - \rho \frac{\partial \varphi}{\partial t}$$

Since solution moves with the sphere

$$\varphi = \varphi(\vec{r} - \vec{u}t, \vec{u})$$

then $\frac{\partial \varphi}{\partial t} = \frac{\partial \varphi}{\partial \vec{u}} \cdot \dot{\vec{u}} - \vec{u} \cdot \operatorname{grad} \varphi$ which gives

$$p = p_0 + \rho u^2 (9 \cos^2\theta - 5)/8 + \rho R \vec{n} \cdot \dot{u}$$

Note, that for uniformly moving sphere $\dot{u} = 0$ and the force $\oint p \cdot dS = 0$