Energy and momentum fluxes

Let us look how the energy density changes with time. Energy per unit volume is $\rho\left(E + \frac{v^2}{2}\right)$

Taking derivative

$$\frac{\partial}{\partial t}\frac{\rho v^2}{2} = \frac{v^2}{2}\frac{\partial \rho}{\partial t} + \rho \vec{v}\cdot\frac{\partial \vec{v}}{\partial t}$$

From continuity equation $\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho\vec{v}) = 0$

and the Euler's equation $\frac{\partial \vec{v}}{\partial t} + (\vec{v}\cdot\vec{\nabla})\vec{v} = -\frac{\nabla p}{\rho}$

we express time derivatives, then

$$\frac{\partial}{\partial t}\rho\frac{v^2}{2} = -\frac{v^2}{2}\operatorname{div}(\rho\vec{v}) - \vec{v}\cdot\vec{\nabla}p - \rho\vec{v}\cdot(\vec{v}\cdot\vec{\nabla})\vec{v}$$

In the last term $\vec{v}\cdot(\vec{v}\cdot\vec{\nabla})\vec{v} = \vec{v}\cdot\frac{\vec{\nabla}v^2}{2}$

Because for enthalpy $dW = Tds + Vdp = Tds + \frac{dp}{\rho} \Rightarrow$

$$dp = \rho dW - \rho T ds \Rightarrow \nabla p = \rho\nabla W - \rho T\nabla S$$

we obtain

$$\frac{\partial}{\partial t}\left(\rho\frac{v^2}{2}\right) = -\frac{v^2}{2}\operatorname{div}(\rho\vec{v}) - \rho\vec{v}\cdot\vec{\nabla}\left(\frac{v^2}{2} + W\right) + \rho T\vec{v}\cdot\nabla S$$

For calculating $\frac{\partial(\rho E)}{\partial t}$ we use

$$dE = T\,dS - p\,dV = T\,dS + \frac{p\,d\rho}{\rho^2}$$

Then

$$d(\rho E) = E\,d\rho + \rho\,dE = E\,d\rho + \frac{P}{\rho}\,d\rho + \rho T\,dS =$$

$$= E\,d\rho + PV\,d\rho + T\,dS = W\,d\rho + \rho T\,dS$$

(Thus $\frac{\partial(\rho E)}{\partial t} = W\frac{\partial \rho}{\partial t} + \rho T\frac{\partial S}{\partial t} = -W\,\mathrm{div}(\rho\vec{v}) - \rho T\vec{v}\cdot\vec{\nabla}S$

Here we used equation of adiabaticity $\frac{dS}{dt} = 0$

Combining both time derivatives entropic terms cancel and we get

$$\frac{\partial}{\partial t}\left(\rho\frac{v^2}{2} + \rho E\right) = -\left(\frac{v^2}{2} + W\right)\mathrm{div}(\rho\vec{v}) - \rho\vec{v}\cdot\vec{\nabla}\left(\frac{v^2}{2} + W\right) \Rightarrow$$

$$\frac{\partial}{\partial t}\left(\rho\frac{v^2}{2} + \rho E\right) = -\,\mathrm{div}\left[\rho\vec{v}\left(\frac{v^2}{2} + W\right)\right]$$

As usual r.h.s is the divergence of the flux.

$$\frac{\partial}{\partial t}\int\left(\rho\frac{v^2}{2}+\rho E\right)dV = -\oint \rho\left(W+\frac{v^2}{2}\right)\vec{v}\cdot d\vec{S}$$

Thus $\rho\vec{v}\left(\frac{v^2}{2}+W\right)$ may be called the <u>energy</u> <u>flux</u> <u>density</u>. Its magnitude is the amount of energy passing in unit time through unit area perpendicular to the velocity.

Note that the energy flux is

$$\rho\vec{v}\left(W+\frac{v^2}{2}\right) = \rho\vec{v}\left(E+\frac{v^2}{2}\right)+p\vec{v}$$

which is not equal to the energy density times $\vec{v}$. Additional term has simple meaning $\oint p\vec{v}\cdot d\vec{S}$ is the work done by pressure forces on the fluid within the surface

That means that for energy there is no conservation law for unit mass $\frac{d(\cdot)}{dt}=0$ that is valid for entropy.

The momentum flux

The momentum of unit volume is $\rho\vec{v}$

Writing its change in components

$$\frac{\partial(\rho v_i)}{\partial t} = \rho \frac{\partial v_i}{\partial t} + \frac{\partial \rho}{\partial t} v_i$$

Using again the continuity equation and the Euler's equation for $\frac{\partial}{\partial t}$ we get

$$\frac{\partial (\rho v_i)}{\partial t} = -\rho v_k \frac{\partial v_i}{\partial x_k} - \frac{\partial p}{\partial x_i} - v_i \frac{\partial(\rho v_k)}{\partial x_k} =$$

$$= -\frac{\partial p}{\partial x_i} - \frac{\partial}{\partial x_k}(\rho v_i v_k)$$

We can rewrite it as

$$\frac{\partial}{\partial t}(\rho v_i) = -\frac{\partial \Pi_{ik}}{\partial x_k}$$

where the tensor $\Pi_{ik} = \Pi_{ki}$ is defined as

$$\Pi_{ik} = p\,\delta_{ik} + \rho v_i v_k$$

Integrating it over volume we obtain (104)

$$\frac{\partial}{\partial t}\int \rho v_i \, dV = -\int \frac{\partial \Pi_{ik}}{\partial x_k} dV = -\oint \Pi_{ik} \, dS_k$$

The tensor Π_{ik} is called <u>momentum flux density tensor</u>.

Energy is a scalar => the energy flux is vector
Momentum is a vector => the momentum flux is tensor of rank two.

Along $\vec{v}$ the flux of parallel momentum is $p+\rho v^2$.

Perpendicular to $\vec{v}$ the flux is just p.