Hydrodynamics = Fluid Mechanics

Basic equations

We study motion of fluids (liquids and gases)
A fluid is regarded as a continuous and homogeneous medium. Any small volume is supposed to be large compared with the distances between the molecules

- The state of moving fluid is determined by the fluid velocity $\vec{v}(\vec{r},t)$ and two thermodynamic quantities, e.g the pressure $p(\vec{r},t)$ and the density $\rho(\vec{r},t)$.

Here $\vec{v}(\vec{r},t)$ is velocity of the fluid at a

- given point $\vec{r}=(x,y,z)$ in space and at a given time t. It is velocity at a fixed point in space, but not of a given fluid particle

The same is true for $p(\vec{r},t)$ and $\rho(\vec{r},t)$

Continuity equation

Conservation of matter

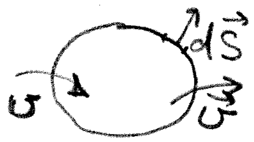
The mass of fluid in the volume V is

$\int \rho \, dV$

Its change per unit time is

- $\frac{\partial}{\partial t} \int \rho \, dV$

The change of the mass is due to the flow of fluid through the surface



$-\oint \rho \vec{\upsilon} \cdot d\vec{S}$ ($d\vec{S}$ is directed outward)

- Then $\frac{\partial}{\partial t} \int \rho \, dV = -\oint \rho \vec{\upsilon} \cdot ds = -\int \operatorname{div}[\rho \upsilon] dV$

$\Rightarrow \int \left(\frac{\partial \rho}{\partial t} + \operatorname{div}[\rho \upsilon] \right) dV = 0 \Rightarrow$

$\frac{\partial \rho}{\partial t} + \operatorname{div} \rho \upsilon = 0$ – continuity equation

$\rho \vec{\upsilon} = \vec{j} \Rightarrow \frac{\partial \rho}{\partial t} + \operatorname{div} \vec{j} = 0$

Euler's equation

The total force acting on a volume V is

$\vec{F} = -\oint p \, d\vec{S}$ (pressure integral over the surface

Transforming it to the volume integral we get

$\vec{F} = -\int \operatorname{grad} p \cdot dV$

- But from the Newton's second law $\vec{F} = m \frac{d\vec{v}}{dt}$

Then per unit volume we obtain

$$\rho \frac{d\vec{v}}{dt} = -\operatorname{grad} p$$

Here $\frac{d\vec{v}}{dt}$ is not the rate of velocity change

- at a fixed point in space but the rate of change of the velocity of a given fluid particle as it moves about in space. We can express it in terms of quantities referring to points fixed in space

$$\vec{v}(\vec{r},t) \qquad \vec{v}(\vec{r}+\vec{dr},t+dt)$$

$$\vec{dr} = \vec{v} \cdot dt$$

$$d\vec{v} = \frac{\partial \vec{v}}{\partial t}\cdot dt + dx\frac{\partial v}{\partial x} + dy\frac{\partial v}{\partial y} + dy\frac{\partial v}{\partial y} \Rightarrow$$

$$\frac{d\vec{v}}{\partial t} = \frac{\partial \vec{v}}{\partial t} + (\vec{v}\cdot\vec{\nabla})\vec{v}$$

As a result the equation of motion is

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v}\cdot\vec{\nabla})\vec{v} = -\frac{\vec{\nabla}p}{\rho} + \vec{f}$$ Euler's Eq. (1755)

- Here we added an external force $\vec{f}$, for gravity $\vec{f} = \vec{g}$

In deriving the equation of motion we didn't take into account energy dissipation, neglecting internal friction (viscosity) and heat exchange.

- Fluid without viscosity and thermal conductivity is called ideal.

The absence of heat exchange means that the motion is adiabatic $\Rightarrow$ entropy of any particle of fluid remains constant $\Rightarrow \frac{ds}{dt} = 0 \Rightarrow$
$\frac{\partial s}{\partial t} + (\vec{v}\cdot\vec{\nabla})s$ or using continuity Eq.

$$\frac{\partial(\rho s)}{\partial t} + \mathrm{div}(\rho s\vec{v}) = 0$$

Boundary conditions

At a fixed boundary $v_n = 0$

At a (moving) boundary between two immiscible fluids $p_1 = p_2$, $v_{n_1} = v_{n_2}$

Hydrostatics

- For a fluid at rest in a gravitational field, Euler's equation takes the form

$$\operatorname{grad} p = \rho \vec{g} \iff \frac{\partial p}{\partial z} = -\rho g$$

For incompressible fluid

$$p = p(0) - \rho g z$$

- For an ideal gas at a homogeneous temperature T we have $pV = NT \Rightarrow p = \frac{\rho T}{m} \Rightarrow$

$$\frac{\partial \rho}{\partial z} = -\frac{\rho g m}{T} \Rightarrow$$

$$\underline{\rho = \rho(0) \exp\left(-\frac{mgz}{T}\right)} \quad \text{Boltzmann's law}$$

For air at $0°C$ $\frac{T}{mg} \simeq 8\,km$

Earth atmosphere is described by neither incompressible nor exponential law because of an inhomogeneous temperature

Assuming a linear temperature decay $T(z) = T_0 - \alpha z$ one gets better approximation

$$\rho(z) = \rho(0)\left(1 - \frac{\alpha z}{T_0}\right)^{mg/\alpha}$$

$\alpha \simeq 6.5$ grad/km

Because $\frac{\partial p}{\partial z} = \rho \vec{g}$ with $g = \text{const}$ then both pressure and density are function of altitude z only. Since p and ρ determine temperature it also should be a function of z only. Different temperatures at the same height produce fluid motion. That is why wind blows in the atmosphere and current flows in the ocean

Another source of atmospheric flow is thermal <u>convection</u> due to a vertical temperature gradient. Suppose that the fluid element V is shifted up adiabatically. For the stable equilibrium it is necessary, that the resulting force brings it back $\Rightarrow$ the element must be heavier than surrounding fluid $V(S(z-dz),z)\Big|_p < V(S(z),z)\Big|_p \Rightarrow$

$$\left(\frac{\partial V}{\partial S}\right)_p \frac{dS}{dz} > 0 .$$

From thermodynamics $\left(\frac{\partial V}{\partial S}\right)_p = \frac{T}{C_p}\left(\frac{\partial V}{\partial T}\right)_p$

Most substances expand on heating $\Rightarrow \left(\frac{\partial V}{\partial S}\right)_p > 0 \Rightarrow$

$\frac{dS}{dT} > 0$. Expanding

$$\frac{dS}{dT} = \left(\frac{\partial S}{\partial T}\right)_p \frac{dT}{dz} + \left(\frac{\partial S}{\partial p}\right)_T \frac{dp}{dz} = \frac{C_p}{T}\frac{dT}{dz} - \left(\frac{\partial V}{\partial T}\right)_p \frac{g}{V} > 0$$

We used $V = \frac{1}{\rho}$, $\frac{\partial p}{\partial z} = -\rho g$.

Thus $-\frac{dT}{dz} < g\beta T / C_p$ where $\beta = \frac{1}{V}\frac{\partial V}{\partial T}\Big|_p$

For the Earth atmosphere the convection threshold is $\simeq 10^\circ/\text{km}$

The simplest motion to consider is isentropic motion where entropy is constant through the volume $S = \text{const.}$

We introduce enthalpy $W = E + pV$, where E is the internal energy

$$dE = TdS - PdV \quad \Rightarrow$$

$$dW = TdS + Vdp$$

Since $S = \text{const} \Rightarrow dW = Vdp = \frac{dp}{\rho}$

and the Euler's equation reads

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \vec{\nabla})\vec{v} = -\nabla W$$

Using $(\vec{v} \cdot \vec{\nabla})\vec{v} = \nabla \frac{v^2}{2} - v \times [\nabla \times v]$ we get

$$\frac{\partial \vec{v}}{\partial t} - \vec{v} \times (\vec{\nabla} \times \vec{v}) = -\vec{\nabla}\left(W + \frac{v^2}{2}\right)$$

Taking rot we obtain equation that contains only velocity

$$\frac{\partial}{\partial t}(\text{rot}\,\vec{v}) = \text{rot}\,(\vec{v} \times \text{rot}\,\vec{v})$$

Bernoulli's equation

Let us consider a steady flow $\frac{\partial v}{\partial t} = 0$

Then $v \times \nabla \times v = \nabla \left(w + \frac{v^2}{2} \right)$ (1)

Let us define streamlines as lines that

$$\frac{dx}{v_x} = \frac{dy}{v_y} = \frac{dz}{v_z}$$

- Tangent to them at any point gives the direction of the velocity at that point.

Then we multiply the equation (1) by the unit vector $\vec{l}$ along the streamline.

$$\vec{l} \cdot \nabla \left(w + \frac{v^2}{2} \right) = \frac{\partial}{\partial l} \left(w + \frac{v^2}{2} \right)$$

- $\vec{l} \cdot (v \times \mathrm{rot}\, v) = 0$ because $v \times \mathrm{rot}\, v \perp \vec{l} \parallel \vec{v}$

As a result we obtain that

$$\frac{\partial}{\partial l} \left(w + \frac{v^2}{2} \right) = 0 \Rightarrow$$

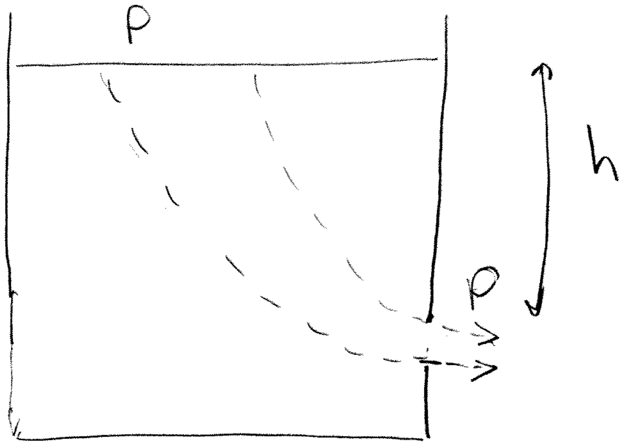
$w + \frac{v^2}{2} = \text{constant}$ along the streamline

This is Bernoulli's equation (1738)

In a gravity field

$$w + gz + \frac{v^2}{2} = const$$

Application, water flowing from the hole



$$W = E + \frac{P}{\rho}, \quad E = const \Rightarrow$$

$$\frac{P}{\rho} + gz + \frac{v^2}{2} = const.$$

Assume pressure at the surface and at the hole is the same. Then we obtain the Torricelli law (1643)

$$v = \sqrt{2gh}$$

40-3 Steady flow—Bernoulli's theorem

Now we want to return to the equation of motion, Eq. (40.8), but limit ourselves to situations in which the flow is "steady." By steady flow we mean that at any one place in the fluid the velocity never changes. The fluid at any point is always replaced by new fluid moving in exactly the same way. The velocity picture always looks the same—$\boldsymbol{v}$ is a static vector field. In the same way that we drew "field lines" in magnetostatics, we can now draw lines which are always tangent to the fluid velocity as shown in Fig. 40-5. These lines are called *streamlines*. For steady flow, they are evidently the actual paths of fluid particles. (In unsteady flow the streamline pattern changes in time, and the streamline pattern at any instant does not represent the path of a fluid particle.)

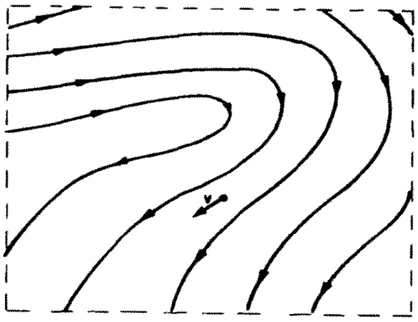


Fig. 40-5. Streamlines in steady fluid flow.

A steady flow does not mean that nothing is happening—atoms in the fluid are moving and changing their velocities. It only means that $\partial\boldsymbol{v}/\partial t = 0$. Then if we take the dot product of $\boldsymbol{v}$ into the equation of motion, the term $\boldsymbol{v}\cdot(\boldsymbol{\Omega}\times\boldsymbol{v})$ drops out, and we are left with

$$\boldsymbol{v}\cdot\boldsymbol{\nabla}\left\{\frac{p}{\rho}+\phi+\frac{1}{2}v^2\right\}=0. \qquad (40.12)$$

This equation says that *for a small displacement in the direction of the fluid velocity* the quantity inside the brackets doesn't change. Now in steady flow all displacements are along streamlines, so Eq (40.12) tells us that *for all the points along a streamline*, we can write

$$\frac{p}{\rho}+\frac{1}{2}v^2+\phi=\text{const (streamline)}. \qquad (40.13)$$

This is *Bernoulli's theorem*. The constant may in general be different for different streamlines; all we know is that the left-hand side of Eq. (40.13) is the same all along a *given streamline*. Incidentally, we may notice that for steady *irrotational* motion for which $\boldsymbol{\Omega} = 0$, the equation of motion (40.8) gives us the relation

$$\boldsymbol{\nabla}\left\{\frac{p}{\rho}+\frac{1}{2}v^2+\phi\right\}=0,$$

so that

$$\frac{p}{\rho}+\frac{1}{2}v^2+\phi=\text{const (everywhere)}. \qquad (40.14)$$

It's just like Eq. (40.13) *except* that *now* the constant has the *same value throughout the fluid.*

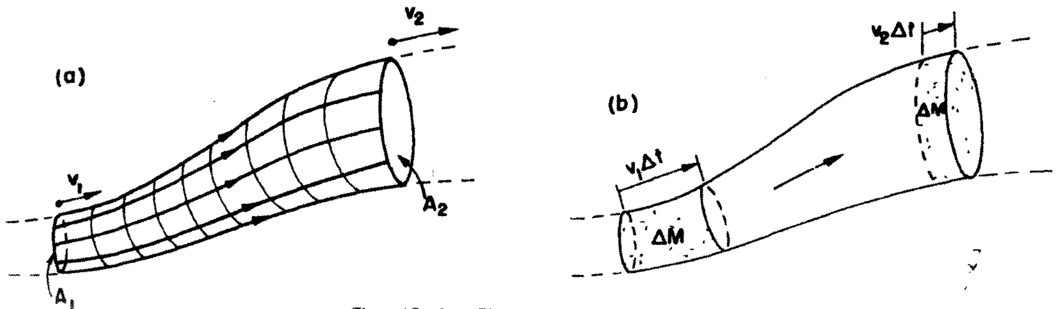


Fig. 40-6. Fluid motion in a flow tube.

The theorem of Bernoulli is in fact nothing more than a statement of the conservation of energy. A conservation theorem such as this gives us a lot of information about a flow without our actually having to solve the detailed equations. Bernoulli's theorem is so important and so simple that we would like to show you how it can be derived in a way that is different from the formal calculations we have just used. Imagine a bundle of adjacent streamlines which form a stream tube as sketched in Fig. 40-6. Since the walls of the tube consist of streamlines, no fluid flows out through the wall. Let's call the area at one end of the stream tube A_1, the fluid velocity there v_1, the density of the fluid ρ_1, and the potential energy ϕ_1. At the other end of the tube, we have the corresponding quantities A_2, v_2, ρ_2, and ϕ_2. Now after a short interval of time Δt, the fluid at A_1 has moved a distance $v_1\,\Delta t$, and the fluid at A_2 has moved a distance $v_2\,\Delta t$ [Fig. 40–6(b)]. The conservation of *mass* requires that the mass which enters through A_1 must be equal to the mass which leaves through A_2. These masses at these two ends must be the same:

$$\Delta M = \rho_1 A_1 v_1\,\Delta t = \rho_2 A_2 v_2\,\Delta t.$$

So we have the equality

$$\rho_1 A_1 v_1 = \rho_2 A_2 v_2. \tag{40.15}$$

This equation tells us that the velocity varies inversely with the area of the stream tube if ρ is constant.

Now we calculate the work done by the fluid pressure. The work done on the fluid entering at A_1 is $p_1 A_1 v_1\,\Delta t$, and the work given up at A_2 is $p_2 A_2 v_2\,\Delta t$ The net work on the fluid between A_1 and A_2 is, therefore,

$$p_1 A_1 v_1\,\Delta t - p_2 A_2 v_2\,\Delta t,$$

which must equal the increase in the energy of a mass ΔM of fluid in going from A_1 to A_2. In other words,

$$p_1 A_1 v_1\,\Delta t - p_2 A_2 v_2\,\Delta t = \Delta M(E_2 - E_1), \tag{40.16}$$

where E_1 is the energy per unit mass of fluid at A_1, and E_2 is the energy per unit mass at A_2. The energy per unit mass of the fluid can be written as

$$E = \tfrac{1}{2}v^2 + \phi + U,$$

where $\frac{1}{2}v^2$ is the kinetic energy per unit mass, ϕ is the potential energy per unit mass, and U is an additional term which represents the internal energy per unit mass of fluid. The internal energy might correspond, for example, to the thermal energy in a compressible fluid, or to chemical energy. All these quantities can vary from point to point. Using this form for the energies in (40.16), we have

$$\frac{p_1 A_1 v_1\,\Delta t}{\Delta M} - \frac{p_2 A_2 v_2\,\Delta t}{\Delta M} = \frac{1}{2}v_2^2 + \phi_2 + U_2 - \frac{1}{2}v_1^2 - \phi_1 - U_1$$

But we have seen that $\Delta M = \rho A v\,\Delta t$, so we get

$$\frac{p_1}{\rho_1} + \frac{1}{2}v_1^2 + \phi_1 + U_1 = \frac{p_2}{\rho_2} + \frac{1}{2}v_2^2 + \phi_2 + U_2, \tag{40.17}$$

which is the Bernoulli result with an additional term for the internal energy. If the fluid is incompressible, the internal energy term is the same on both sides, and we get again that Eq. (40.14) holds along any streamline.

We consider now some simple examples in which the Bernoulli integral gives us a description of the flow. Suppose we have water flowing out of a hole near the bottom of a tank, as drawn in Fig. 40–7. We take a situation in which the flow speed v_{out} at the hole is much larger than the flow speed near the top of the tank; in other words, we imagine that the diameter of the tank is so large that we can neglect the drop in the liquid level. (We could make a more accurate calculation if we wished.) At the top of the tank the pressure is p_0, the atmospheric pressure, and the pressure at the sides of the jet is also p_0. Now we write our Bernoulli equation for a streamline, such as the one shown in the figure. At the top of the tank, we take v equal to zero and we also take the gravity potential ϕ to be zero. At the speed v_{out}, and $\phi = -gh$, so that

$$p_0 = p_0 + \tfrac{1}{2}\rho v_{\text{out}}^2 - \rho gh,$$

or

$$v_{\text{out}} = \sqrt{2gh}. \tag{40.18}$$

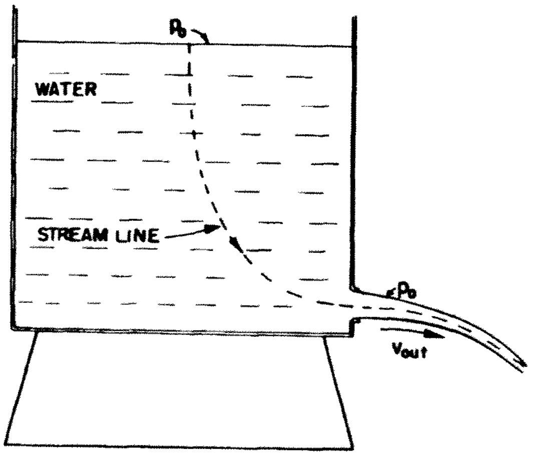


Fig. 40–7. Flow from a tank.

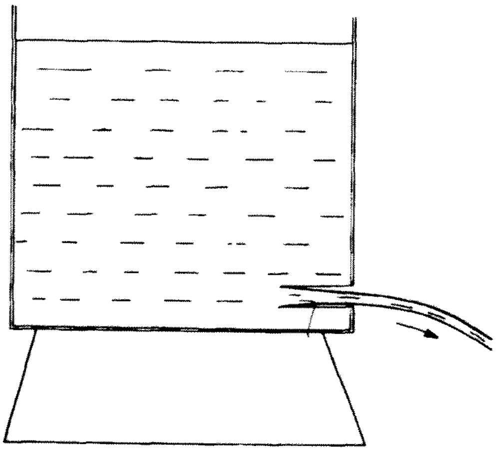


Fig. 40–8. With a re-entrant discharge tube, the stream contracts to one-half the area of the opening.

This velocity is just what we would get for something which falls the distance h. It is not too surprising, since the water at the exit gains kinetic energy at the expense of the potential energy of the water at the top. Do not get the idea, however, that you can figure out the rate that the fluid flows out of the tank by multiplying this velocity by the area of the hole. The fluid velocities as the jet leaves the hole are not all parallel to each other but have components inward toward the center of the stream—the jet is converging. After the jet has gone a little way, the contraction stops and the velocities do become parallel. So the total flow is the velocity times the area *at that point*. In fact, if we have a discharge opening which is just a round hole with a sharp edge, the jet contracts to 62 percent of the area of the hole. The reduced effective area of the discharge varies for different shapes of discharge tubes, and experimental contractions are available as tables of *efflux coefficients*.

If the discharge tube is re-entrant, as shown in Fig. 40–8, it is possible to prove in a most beautiful way that the efflux coefficient is exactly 50 percent. We will give just a hint of how the proof goes. We have used the conservation of energy to get the velocity, Eq. (40.18), but there is also momentum conservation to consider. Since there is an outflow of momentum in the discharge jet, there must be a force applied over the cross section of the discharge tube. Where does the force come from? The force must come from the pressure on the walls. As long as the efflux hole is small and away from the walls, the fluid velocity near the walls of the tank will be very small. Therefore, the pressure on every face is almost exactly the same as the static pressure in a fluid at rest—from Eq. (40.14). Then the static pressure at any point on the side of the tank must be matched by an equal pressure at the point on the opposite wall, *except* at the points on the wall opposite the charge tube. If we calculate the momentum poured out through the jet by this pressure, we can show that the efflux coefficient is 1/2. We cannot use this method for a discharge hole like that shown in Fig. 40–7, however, because the velocity increase along the wall right near the discharge area gives a pressure fall which we are not able to calculate.

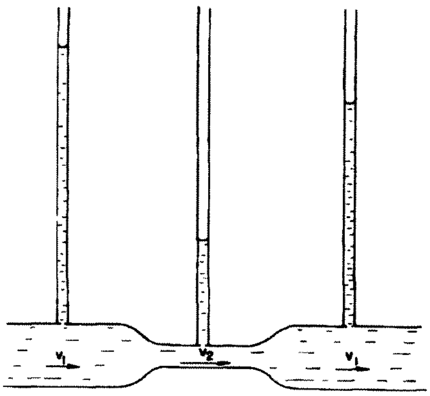


Fig. 40–9. The pressure is lowest where the velocity is highest.

$\frac{\text{momentum}}{\text{time}} = \rho v v S' =$

force $= p S = \rho g h S$

because from Bernoulli's law

$v^2 = 2gh \Rightarrow S' = \frac{1}{2} S$

Let's look at another example—a horizontal pipe with changing cross section, as shown in Fig. 40–9, with water flowing in one end and out the other. The conservation of energy, namely Bernoulli's formula, says that the pressure is lower in the constricted area where the velocity is higher. We can easily demonstrate this effect by measuring the pressure at different cross sections with small vertical columns of water attached to the flow tube through holes small enough so that they do not disturb the flow. The pressure is then measured by the height of water in these vertical columns. The pressure is found to be less at the constriction than it is on either side. If the area beyond the constriction comes back to the same value it had before the constriction, the pressure rises again.

Bernoulli's formula would predict that the pressure downstream of the constriction should be the same as it was upstream, but actually it is noticeably less. The reason that our prediction is wrong is that we have neglected the frictional, viscous forces which cause a pressure drop along the tube. Despite this pressure drop the pressure is definitely lower at the constriction (because of the increased speed) than it is on either side of it—as predicted by Bernoulli. The speed v_2 must certainly exceed v_1 to get the same amount of water through the narrower tube. So the water accelerates in going from the wide to the narrow part. The force that gives this acceleration comes from the drop in pressure.

We can check our results with another simple demonstration. Suppose we have on a tank a discharge tube which throws a jet of water upward as shown in Fig. 40–10. If the efflux velocity were exactly $\sqrt{2gh}$, the discharge water should rise to a level even with the surface of the water in the tank. Experimentally, it falls somewhat short. Our prediction is roughly right, but again viscous friction which has not been included in our energy conservation formula has resulted in a loss of energy.

Have you ever held two pieces of paper close together and tried to blow them apart? Try it! They come *together*. The reason, of course, is that the air has a higher speed going through the constricted space between the sheets than it does when it gets outside. The pressure between the sheets is *lower* than atmospheric pressure, so they come together rather than separating.

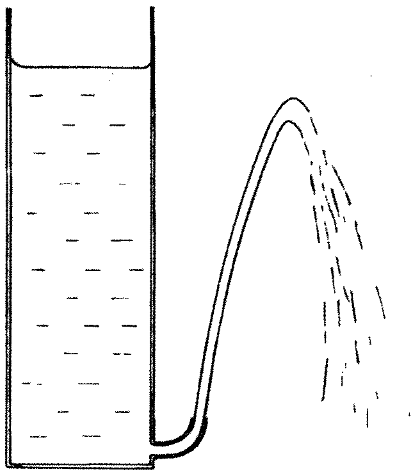


Fig. 40–10. Proof that v is not equal to $\sqrt{2gh}$.

40–4 Circulation

We saw at the beginning of the last section that if we have an incompressible fluid with no circulation, the flow satisfies the following two equations:

$$\nabla \cdot \boldsymbol{v} = 0, \qquad \nabla \times \boldsymbol{v} = 0. \tag{40.19}$$

They are the same as the equations of electrostatics or magnetostatics in empty space. The divergence of the electric field is zero when there are no charges, and the curl of the electrostatic field is always zero. The curl of the magnetic field is zero if there are no currents, and the divergence of the magnetic field is always zero. Therefore, Eqs. (40.19) have the same solutions as the equations for $\boldsymbol{E}$ in electrostatics or for $\boldsymbol{B}$ in magnetostatics. As a matter of fact, we have already solved the problem of the flow of a fluid past a sphere, as an electrostatic analogy, in Section 12–5. The electrostatic analog is a uniform electric field plus a dipole field. The dipole field is so adjusted that the flow velocity normal to the surface of the sphere is zero. The same problem for the flow past a cylinder can be worked out in a similar way by using a suitable line dipole with a uniform flow field. This solution holds for a situation in which the fluid velocity at large distances is constant—both in magnitude and direction. The solution is sketched in Fig. 40–11(a).

There is another solution for the flow around a cylinder when the conditions are such that the fluid at large distances moves in circles around the cylinder. The flow is, then, circular everywhere, as in Fig. 40–11(b). Such a flow has a circulation around the cylinder, although $\nabla \times \boldsymbol{v}$ is still zero *in the fluid*. How can there be circulation without a curl? We have a circulation around the cylinder because the line integral of $\boldsymbol{v}$ around any loop *enclosing* the cylinder is not zero. At the same time, the line integral of $\boldsymbol{v}$ around any closed path which does *not* include the cylinder is zero. We saw the same thing when we found the magnetic field around a wire. The curl of $\boldsymbol{B}$ was zero outside of the wire, although a line integral of $\boldsymbol{B}$ around a path which encloses the wire did not vanish. The velocity field in an irrotational circulation around a cylinder is precisely the same as the magnetic field around a wire. For a circular path with its center at the center of the cylinder, the line integral of the velocity is

$$\oint \boldsymbol{v} \cdot d\boldsymbol{s} = 2\pi r v.$$

For irrotational flow the integral must be independent of r. Let's call the constant

(a)

(b)

(c)

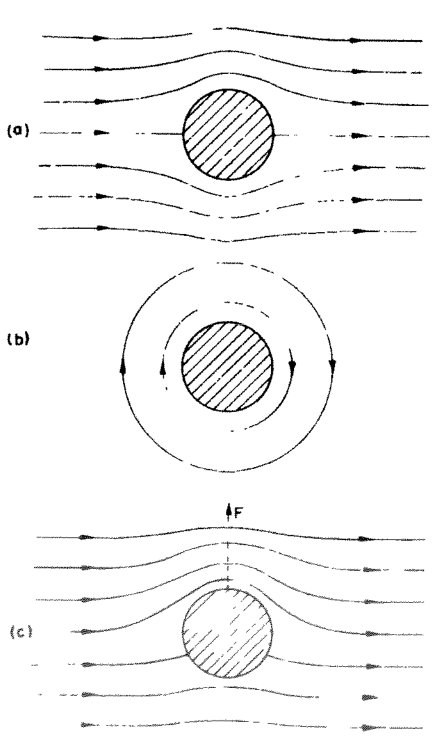


Fig. 40–11. (a) Ideal fluid flow past a cylinder. (b) Circulation around a cylinder. (c) The superposition of (a) and (b).