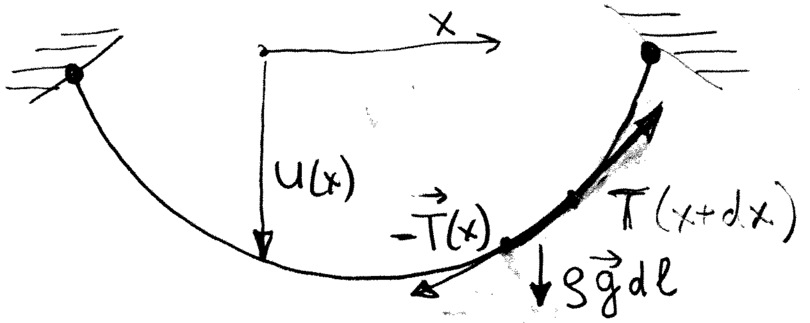
Catenary

Consider a hanging flexible chain or cable.
Suspend it from the two ends
What is the shape of the chain?

R. Hooke (1670)
Leibniz (1691)
Huygens
Johann Bernoulli



It is called Catenary

Let us write the force balance equation for the little piece of the chain dl

$$\vec{T}(x+dx) - \vec{T}(x) = \rho\vec{g}\,dl \qquad (1)$$

Here ρ is mass density (per unit length)

Since $dl = \sqrt{dx^2 + du^2} = dx\sqrt{1+u'^2}$

we can rewrite force balance as

$$\frac{d\vec{T}}{dx} = \rho\vec{g}\sqrt{1+u'^2}$$

Since $\vec{g}$ is along the y-axis we have

$$\frac{dT_x}{dx} = 0 \quad \Rightarrow \quad T_x = \text{const}$$

$$\frac{dT_y}{dx} = \rho g \sqrt{1+u'^2}$$

Tension $\vec{T}$ is tangential to the line $\Rightarrow$

$$\frac{T_y}{T_x} = \frac{du}{dx}$$

And we get differential equation

$$u'' = \frac{\rho g}{T_x} \sqrt{1+u'^2}$$

Substituting $u' = y$, $\frac{\rho g}{T_x} = a$

$$y' = a\sqrt{1+y^2}$$

Since $\mathrm{sh}'x = \mathrm{ch}\,x = \sqrt{1+\mathrm{sh}^2 x}$, then

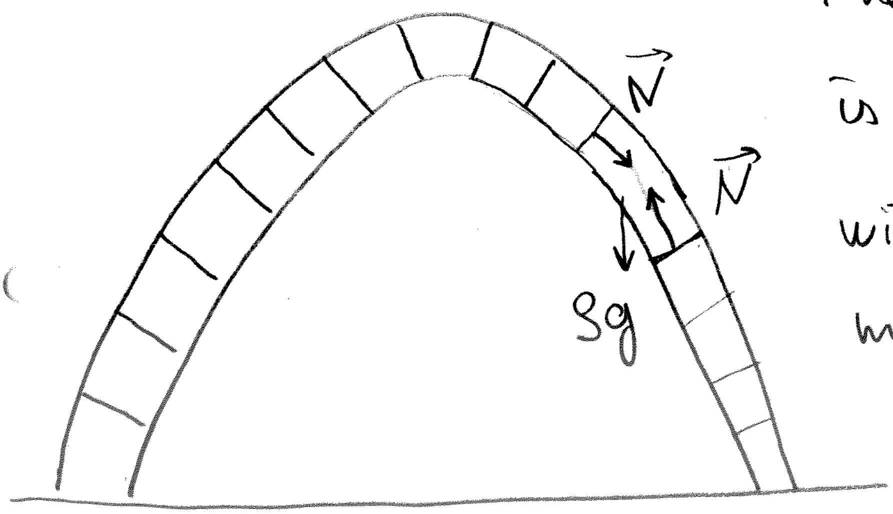
$y = \mathrm{sh}(ax)$ and

$$u(x) = \frac{T_x}{\rho g}\left(\mathrm{ch}\left(\frac{\rho g}{T_x}x\right) - 1\right).$$

We fixed the integration constant by the condition that at the symmetry axis $u(x=0) = 0$

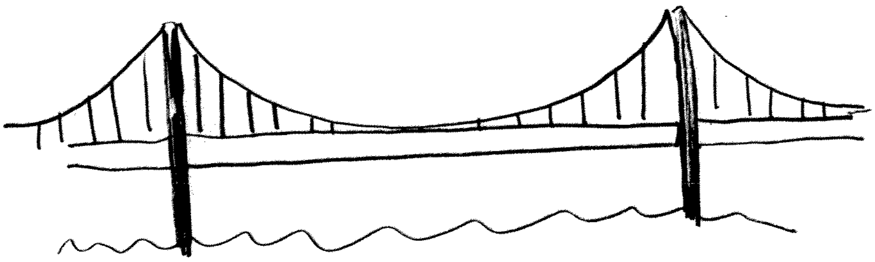
T_x should be fixed by the boundary conditions (chain length).

Inverted catenary arch is the ideal curve for an arch which supports only its own weight



The deformation is pure compression without bending moment

Suspension bridges



If the mass of cables is negligible compared to the mass of the road deck, then Eq. (1) should be replaced by

$$\vec{T}(x+dx) - \vec{T}(x) = \rho_0 \vec{g}\, dx \quad \Rightarrow$$

$$T_x = \text{const}$$

$$\frac{dT_y}{dx} = \rho_0 g \quad \Rightarrow \quad T_y = \rho_0 g x$$

and $u' = \dfrac{T_y}{T_x} \Rightarrow$

$$u = \frac{\rho_0 g}{T_x} \frac{x^2}{2} \quad \text{— parabola!}$$

Elastic string

Consider elastic string under tension T_0

If we deform the string then its length will change by

$$\delta L = \int \sqrt{1+u'^2}\, dx$$

Change in energy is

$$\delta E_{el} = T_0 \cdot \delta L = T_0 \int \sqrt{1+u'^2}\, dx \approx$$

$$\approx T_0 \int \frac{u'^2}{2}\, dx. \quad + \text{const}$$

In gravitational field

$$\delta E = \int \left(T_0 \frac{u'^2}{2} + \rho g u \right) dx$$

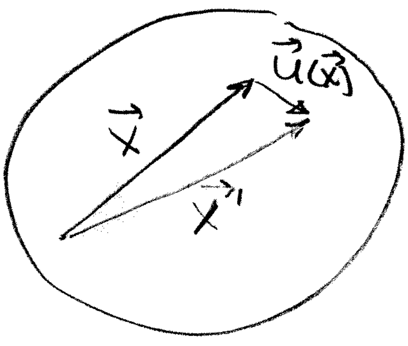
minimizing it we obtain

$$T_0 u'' = \rho g \quad \text{and} \quad u = \frac{\rho g}{T_0} \frac{x^2}{2}$$

Compare with previous results!

Elasticity theory

Under the action of applied forces, solid bodies deform, i.e. they change in shape and volume. To describe the deformation, consider some particular point. Let its radius vector before the deformation be $\vec{x}$, and after the deformation have a different value $\vec{x}'$. Then the displacement



of this point is given by the displacement vector

$$\vec{u}(x) = \vec{x}' - \vec{x}, \quad u_i = x_i' - x_i$$

Under the deformation the distances between points of the body change, $dx_i' = dx_i + du_i$

If before the deformation $dl^2 = \sum_i dx_i^2$, then after $dl'^2 = \sum_i dx_i'^2 = \sum_i (dx_i + du_i)^2$

Using $du_i = \frac{\partial u_i}{\partial x_k} dx_k$ we obtain

$$dl'^2 = dl^2 + 2\frac{\partial u_i}{\partial x_k} dx_i dx_k + \frac{\partial u_i}{\partial x_k}\frac{\partial u_i}{\partial x_l} dx_k dx_l$$

(we sum over repeating indices)

Changing indices under summation we can transform the last expression into

$$dl'^2 = dl^2 + 2u_{ik}\,dx_i\,dx_k$$ with

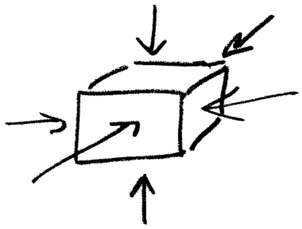
$$u_{ik} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i} + \frac{\partial u_l}{\partial x_i}\frac{\partial u_l}{\partial x_k}\right)$$ – strain tensor

$$u_{ik} = u_{ki}$$

Usually $\frac{\partial u_i}{\partial x_k} \ll 1$ and we can neglect the quadratic term in u_{ik}, then $u_{ik} = \frac{1}{2}\left[\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i}\right]$

We can diagonolize u_{ik} at any given point

$$\begin{pmatrix} u_{11} & 0 & 0 \\ 0 & u_{22} & 0 \\ 0 & 0 & u_{33} \end{pmatrix}$$



Then $dx_1' = (1+u_{11})dx_1$, $dx_2' = (1+u_{22})dx_2$, $dx_3' = \ldots$

The volume $dV = dx_1 dx_2 dx_3 \Rightarrow$

$$dV' = dV(1+u_{11})(1+u_{22})(1+u_{33}) = dV(1+\sum_i u_{ii})$$

$$dV' = dV(1+u_{ii})$$

Since the trace u_{ii} is invarient, the relative volume change is

$$\frac{dV' - dV}{dV} = u_{ii} = \operatorname{div}\vec{u}$$

Stress tensor

Before the deformation body is in equilibrium. $\Rightarrow$ After deformation forces arise which tend to return the body to equilibrium. These forces are called <u>internal stresses</u>.

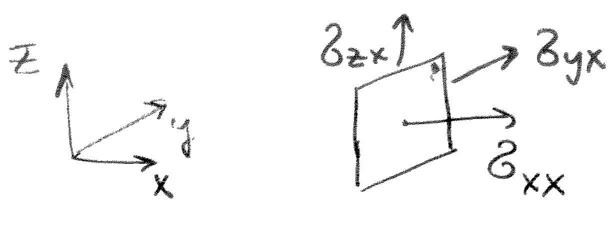
It is important that the internal stresses are due to molecular forces which have very short range of action. Thus in macroscopic elasticity theory these forces may be considered as having zero range of action. (we do not consider here piezoelectrics).

Consider force acting on some volume  $\int \vec{F} dV$

The total force from the inner part is zero (Newton's third law). Thus the total force is the force from the neighbouring parts of the body which can be considered as applied to the surface $\Rightarrow$

$$\int \vec{F}\, dV = \oint \sigma_{ik}\, dS_k \Rightarrow F_i = \frac{\partial \sigma_{ik}}{\partial x_k}$$

σ_{ik} – <u>stress tensor</u>



Moment of the force (Torque) can be written as

$$M = \int (F_i x_k - F_k x_i)\, dV =$$

$$= \int \left(\frac{\partial \sigma_{il}}{\partial x_l} x_k - \frac{\partial \sigma_{kl}}{\partial x_l} x_i \right) dV =$$

$$= \int \frac{\partial (\sigma_{il} x_k - \sigma_{kl} x_i)}{\partial x_l}\, dV - \int \left(\sigma_{il} \frac{\partial x_k}{\partial x_l} - \sigma_{kl} \frac{\partial x_i}{\partial x_l} \right) dV =$$

$$= \oint (\sigma_{il} x_k - \sigma_{kl} x_i)\, dS_l - \int (\sigma_{ik} - \sigma_{ki})\, dV$$

(we used $\frac{\partial x_i}{\partial x_k} = \delta_{ik}$)

The torque will be an integral over the surface if the stress tensor is symmetrical

$$\underline{\sigma_{ik} = \sigma_{ki}}$$

Hydrostatic compression $-p\, dS_i = -p\, \delta_{ik}\, dS_k \Rightarrow$

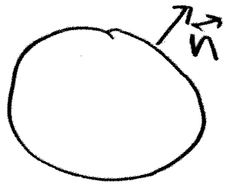
$$\sigma_{ik} = -p\, \delta_{ik}$$

In equilibrium $\vec{F_i} = 0 \Rightarrow \frac{\partial \sigma_{ik}}{\partial x_k} = 0$

If we have external bulk forces (e.g. gravitation)

$$\frac{\partial \sigma_{ik}}{\partial x_k} + \rho g_i = 0$$

If external forces are applied at the surface



$$P_i\, dS = \sigma_{ik}\, dS_k = \sigma_{ik} n_k\, dS \Rightarrow$$

boundary conditions $\sigma_{ik} n_k = P_i$

The work done (by the internal stresses) under deformation

$$\int \delta R\, dV = \int F_i\, \delta u_i\, dV = \int \frac{\partial \sigma_{ik}}{\partial x_k}\, \delta u_i\, dV =$$

Integrating by parts $= \oint \sigma_{ik}\, \delta u_i\, dS_k - \int \sigma_{ik} \frac{\partial \delta u_i}{\partial x_k}\, dV$

Taking surface in the first integral to infinity where there are no deformation and using $\sigma_{ik} = \sigma_{ki}$ we obtain

$$\int \delta R\, dV = - \int \sigma_{ik}\, \delta u_{ik}\, dV \Rightarrow$$

The change in the internal energy

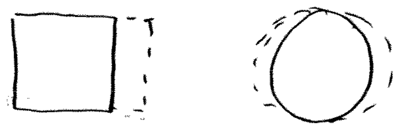
$$d\mathcal{E} = T\,dS - dR = T\,dS + \sigma_{ik}\, du_{ik}\,,\ \text{then}$$

$$dF = d(\mathcal{E} - TS) = -S\,dT + \sigma_{ik}\, du_{ik} \quad \text{and}$$

$$\sigma_{ik} = \left(\frac{\partial F}{\partial u_{ik}}\right)_T .$$

Examples of deformations

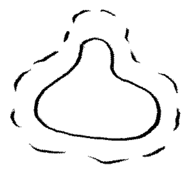
Unilateral compression



$$u_x = \varepsilon x \,, \quad u_y = u_z = 0$$

$$u_{ik} = \begin{pmatrix} \varepsilon & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

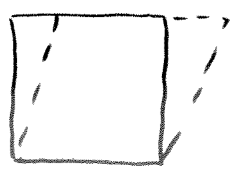
Hydrostatic compression (preserves shape)



$$u_i = \varepsilon x_i$$

$$u_{ik} = \begin{pmatrix} \varepsilon & 0 & 0 \\ 0 & \varepsilon & 0 \\ 0 & 0 & \varepsilon \end{pmatrix}$$

Shear (preserves volume)



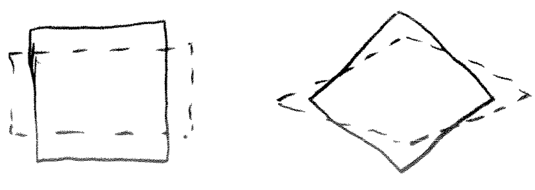
$$u_x = 2\varepsilon y$$

$$u_y = u_z = 0$$

$$u_{ik} = \begin{pmatrix} 0 & \varepsilon & 0 \\ \varepsilon & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Or $u_x = \varepsilon x$

$$u_y = -\varepsilon y$$



$$u_{ik} = \begin{pmatrix} \varepsilon & 0 & 0 \\ 0 & -\varepsilon & 0 \\ 0 & 0 & 0 \end{pmatrix}$$