Exercise 1. ***Third-order perturbation theory***

Assuming a non degenerate energy spectrum, compute the third-order correction to the energy eigenvalues in the framework of time-independent perturbation theory.

Exercise 2. ***Exact solution vs perturbation theory***

Consider the following 2×2 Hamiltonian for a 2-state system:

$$H = H_0 + \lambda V$$

with

$$H_0 = \begin{pmatrix} E_1^{(0)} & 0 \\ 0 & E_2^{(0)} \end{pmatrix}, \qquad V = \begin{pmatrix} 0 & v \\ v & 0 \end{pmatrix}.$$

(a) Solve exactly the problem

$$H\,\Psi = E\,\Psi,$$

determining the eigenvalues and the eigenvectors.

(b) Using time independent perturbation theory in λ derive the **first**-order correction to the eigenvectors, and the **second**-order correction to the eigenvalues.

(c) Expand the exact result (a) in λ and compare them with what you obtained in (b).

Exercise 3. ***Quasi-degenerate energy levels***

In this exercise we want to see what happens when two energy levels are almost equal. For this consider two quasi-degenerate energy levels of the unperturbed Hamiltonian H_0

$$E_1^{(0)} = E^{(0)} + \epsilon, \quad E_2^{(0)} = E^{(0)} - \epsilon, \quad \text{with } \epsilon \text{ small.}$$

We want to solve the Schrödinger equation $(H_0 + \lambda V)\,|\psi_n\rangle = E_n\,|\psi_n\rangle$ perturbatively for $n = 1, 2$. Decompose $|\psi_n\rangle$ as follows

$$|\psi_n\rangle = \sum_k |\psi_k^{(0)}\rangle\,\langle\psi_k^{(0)}|\psi_n\rangle \equiv \sum_k a_{nk}\,|\psi_k^{(0)}\rangle\,.$$

(a) Following the steps of the derivation of the determinant condition for the degenerate case and neglecting terms less singular than ϵ^{-1} show that

$$\det\begin{pmatrix} E_1^{(0)} + \lambda V_{11} - E_n & \lambda V_{12} \\ \lambda V_{21} & E_2^{(0)} + \lambda V_{22} - E_n \end{pmatrix} = 0, \quad n = 1, 2. \tag{1}$$

(b) Solve the above equation for E_n to get

$$E_{1,2} = \bar{E} \pm \left\{ (\Delta E)^2 + \lambda^2 |V_{12}|^2 \right\}^{1/2} ,$$

where

$$\bar{E} = \frac{1}{2} \left(E_1^{(0)} + \lambda V_{11} + E_2^{(0)} + \lambda V_{22} \right) ,$$
$$\Delta E = \frac{1}{2} \left(E_1^{(0)} + \lambda V_{11} - E_2^{(0)} - \lambda V_{22} \right) .$$

(c) Can the perturbation make the energy levels cross ?

(d) Show that equation (1) reduces to the degenerate case in the limit $\epsilon \to 0$.